The Ordered Set $\mathbb{R}$ — Math 323

Orderings

- An ***ordering*** on a set S is a binary predicate "$<$" on S satisfying two properties:
  - [Trichotomy] $\forall\, x, y \in S$, one and only one of the following holds: $x < y$; $y < x$; or $x = y$.
  - [Transitivity] $\forall\, x, y, z \in S$, $x < y$ and $y < z \implies x < z$.

  From this relation, we can build the standard set of comparison relations: $x > y$ means $y < x$;
  $x \le y$ means $[x < y \text{ or } x = y]$;
  etc.
- An ***ordered set*** is a set S equipped with an ordering $<$ on S.
- An ***ordered field*** is a *field* F equipped with an ordering $<$ on F that is compatible with F's field operations, in the following sense: $x > 0$ and $y > 0 \implies x + y > 0$ and $xy > 0$.
  - e.g., the fields $\mathbb{R}$ or $\mathbb{Q}$, equipped with the usual orderings, are ordered fields;
    however, no matter how we define an ordering on the *set* $\mathbb{C}$, it cannot be made into an ordered field.

Upper and lower bounds for a set $A \subset \mathbb{R}$

- u is an ***upper bound*** for A means $\forall\, a \in A,\ a \le u$;
  A is ***bounded above*** if A has an upper bound — i.e., if $\exists\, u \in \mathbb{R}$ such that $\forall\, a \in A,\ a \le u$.
- Similarly, ℓ is a ***lower bound*** for A means $\forall\, a \in A,\ a \ge \ell$;
  A is ***bounded below*** if A has a lower bound — i.e., if $\exists\, \ell \in \mathbb{R}$ such that $\forall\, a \in A,\ a \ge \ell$.
- A is ***bounded*** if A is both bounded above and bounded below — i.e., if $\exists\, \ell, u \in \mathbb{R}$ such that $\forall\, a \in A,\ \ell \le a \le u$.

Note that upper and lower bounds for a set A must be real numbers, but they need not be elements of A.

The supremum and infimum of a set $A \subset \mathbb{R}$

- The ***supremum*** of A, $\sup A$, is the least upper bound for A.
  - $\sup A = +\infty$ if A is not bounded above; $\sup \varnothing = -\infty$.
  - Formally, $s = \sup A$ is an upper bound for A such that every upper bound u for A satisfies $u \ge s$.
  - Characterization of the supremum: $s = \sup A \iff$ (i) $\forall\, a \in A,\ a \le s$, and
    (ii) $\forall\, x < s,\ \exists\, a \in A$ with $a > x$.
  - Proving inequalities involving the supremum:
    - To show that $x \ge \sup A$, simply show that x is an upper bound for A.
    - We generally show that $x \le \sup A$, by showing that $\forall\, \alpha > \sup A,\ x \le \alpha$.
- The ***infimum*** of A, $\inf A$, is the greatest lower bound for A.
  - $\inf A = -\infty$ if A is not bounded below; $\inf \varnothing = +\infty$.
  - Formally, $i = \inf A$ is a lower bound for A such that every lower bound ℓ for A satisfies $\ell \le i$.
  - Characterization of the infimum: $i = \inf A \iff$ (i) $\forall\, a \in A,\ a \ge i$, and
    (ii) $\forall\, x > i,\ \exists\, a \in A$ with $a < x$.
  - Proving inequalities involving the infimum:
    - To show that $x \le \inf A$, simply show that x is a lower bound for A.
    - We generally show that $x \ge \inf A$, by showing that $\forall\, \alpha < \inf A,\ x \ge \alpha$.

Note that every set $A \subset \mathbb{R}$ has both a supremum and an infimum — if not as a real number, then as $+\infty$ or $-\infty$;
but as with upper and lower bounds, the infimum and supremum need not be elements of A.

Important properties of the supremum and infimum:

- If $A \subset \mathbb{R}$ is nonempty and bounded above, then $\sup A$ exists in $\mathbb{R}$. [Order-completeness of $\mathbb{R}$]
  - Note that $\mathbb{Q}$ is not order-complete!
- $\sup(-A) = -\inf A$, where $-A \overset{\text{def}}{=} \{\, -a \,:\, a \in A \,\}$.
  - Consequently, via order-completeness of $\mathbb{R}$, if $A \subset \mathbb{R}$ is nonempty and bounded below, then $\inf A$ exists in $\mathbb{R}$.
- The supremum and infimum do not respect strict inequality, only $\ge$ and $\le$.
  - e.g., all that we can conclude from "$\forall\, a \in A,\ a < x$" is that $\sup A \le x$.

A ***sequence*** in a set S is, informally, an infinite, ordered list of objects $s_1,\ s_2,\ s_3,\ \ldots \in S$.
[Formally, a sequence is S is defined to be a function from $\mathbb{N} \to S$, though this is seldom how we conceptualize it.]

Notation for sequences mirrors that for sets, but we use ( ) rather than { } to indicate that order is significant.

- One way of denoting a sequence is by listing its ***terms***, as in $(s_1,\ s_2,\ s_3,\ \ldots)$.
- We can avoid the ellipses by allowing $\mathbb{N}$ to carry the order information, as in $(s_n\ :\ n \in \mathbb{N})$.
  - Here, n is called the ***index variable*** and $\mathbb{N}$ the ***index set***; of course, any other letter could be used instead of n.
  - This is often shortened to $(s_n)_{n\in\mathbb{N}}$, or even simply (s_n) when the index variable is clear from context.

Convergence and divergence of a sequence (a_n) in $\mathbb{R}$

To properly consider the convergence or divergence of a sequence, we observe what happens to the set of all terms of a sequence as we toss out more and more of its initial terms. An important aspect of convergence and divergence is that the first ten, hundred, or million terms thus have no impact on the eventual outcome of this process.

- We say that (a_n) ***converges to*** $x \in \mathbb{R}$, written $(a_n) \to x$, if as we toss out more and more of its initial terms, the rest pinch closer and closer to x.
  - Definition: $(a_n) \to x$ means $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ so that $\forall\, n \geq N,\ |a_n - x| < \varepsilon$.
  - If (a_n) converges to x, we call x its ***limit*** and write $\lim_{n\to\infty} a_n = x$.
  - We say that (a_n) ***converges*** if $\exists\, x \in \mathbb{R}$ for which $(a_n) \to x$.
  - Limits respect non-strict inequality:
    - If (a_n) converges and $\forall\, n \in \mathbb{N},\ a_n \leq u$, then $\lim_{n\to\infty} a_n \leq u$.
    - If (a_n) converges and $\forall\, n \in \mathbb{N},\ a_n \geq \ell$, then $\lim_{n\to\infty} a_n \geq \ell$.

    Note well that limits do not, however, respect *strict* inequality; e.g., $\forall\, n \in \mathbb{N},\ \frac{1}{n} > 0$, but $\lim_{n\to\infty} \frac{1}{n} = 0$.
- Any sequence that doesn't converge (i.e., doesn't converge to x for any $x \in \mathbb{R}$) is said to ***diverge***.
  Two particular sorts of divergence are particularly important:
  - Definition: $(a_n) \to +\infty$ means $\forall\, M \in \mathbb{R},\ \exists\, N \in \mathbb{N}$ so that $\forall\, n \geq N,\ a_n > M$.
    - i.e., as we toss out more and more of its initial terms, the rest push ever farther toward the right end of $\mathbb{R}$.
  - Definition: $(a_n) \to -\infty$ means $\forall\, M \in \mathbb{R},\ \exists\, N \in \mathbb{N}$ so that $\forall\, n \geq N,\ a_n < M$.
    - i.e., as we toss out more and more of its initial terms, the rest push ever farther toward the left end of $\mathbb{R}$.

Limits supremum and infimum of a sequence (a_n) in $\mathbb{R}$

We define boundedness, supremum, and infimum of a sequence to be the corresponding notions for its *set* of terms; while these can be useful, they (unlike convergence and divergence) are sensitive to the initial terms of a sequence. For this reason, we make two more powerful definitions:

- The ***limit supremum*** of (a_n), is defined by $\limsup\limits_{n\to\infty} a_n \overset{\text{def}}{=} \inf\limits_{n\geq 1}\left[\sup\limits_{k\geq n} a_k\right]$.
  - Characterization of $\limsup_{n\to\infty} a_n$:

$$x = \limsup_{n\to\infty} a_n \iff \begin{array}{l}\text{(i) } \forall\, \alpha > x,\ a_n > \alpha \text{ for only finitely many } n \in \mathbb{N};\ \text{and} \\ \text{(ii) } \forall\, \alpha < x,\ a_n > \alpha \text{ for infinitely many } n \in \mathbb{N}.\end{array}$$

- The ***limit infimum*** of (a_n), is defined by $\liminf\limits_{n\to\infty} a_n \overset{\text{def}}{=} \sup\limits_{n\geq 1}\left[\inf\limits_{k\geq n} a_k\right]$.
  - Characterization of $\liminf_{n\to\infty} a_n$:

$$x = \liminf_{n\to\infty} a_n \iff \begin{array}{l}\text{(i) } \forall\, \alpha < x,\ a_n < \alpha \text{ for only finitely many } n \in \mathbb{N};\ \text{and} \\ \text{(ii) } \forall\, \alpha > x,\ a_n < \alpha \text{ for infinitely many } n \in \mathbb{N}.\end{array}$$

Note that the $\limsup$ and $\liminf$ are insensitive to the initial terms of a sequence (like the limit) but always exist (like the supremum and infimum), at least in the sense of $+\infty$ or $-\infty$ due to the fact that they are defined purely in terms of inf's and sup's. Moreover, they nicely generalize convergence and divergence:

- $(a_n) \to [$ any of x, $+\infty$, or $-\infty]$ $\iff$ both $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ give that result.

Equivalence

A set with no additional structure possesses no identifying characteristics besides than its elements; if we go one step further and don't concerns ourselves with what those elements are, but just "how many" there are (which might not be a number!), then we arrive at the most fundamental sense of equivalence of sets.

- An ***equivalence relation*** is a binary predicate $\sim$ (on some type of object) for which three conditions hold: (i) $x \sim x \quad [\forall\, x]$
  (ii) $x \sim y \implies y \sim x \quad [\forall\, x, y]$
  (iii) $x \sim y$ and $y \sim z \implies x \sim z \quad [\forall\, x, y, z]$
  - An equivalence relation gives us a way of [re]defining what it means for two objects to be the "same"; in other words, equivalence relations generalize the notion of equality.
  - Some examples of well-known equivalence relations are equality of numbers and equality of sets (indeed, these are our prototypes).
- We call two sets A and B ***equivalent***, written $A \sim B$, if there exists a bijective function $f : A \to B$.
  - Note that this is an equivalence relation on sets (outline of proof: use the identity function, the inverse, and the composition, respectively, and check bijectivity).
  - If $A \sim B$, we often write $|A| = |B|$ and say that A and B have the same ***cardinality***.
  - Theorem [Schroeder-Bernstein]: if $\exists$ injective functions $f : A \to B$ and $g : B \to A$, then $A \sim B$.

Finite and infinite sets

- We call a set A...
  - ***finite*** if $A = \varnothing$ or $\exists\, N \in \mathbb{N}$ for which $A \sim \{\, 1, 2, \ldots, n \,\}$, or
  - ***infinite*** if it is not finite; more specifically, ***countably infinite*** if $A \sim \mathbb{N}$, or ***uncountable*** if $A \nsim \mathbb{N}$.
  - ***countable*** if it is either finite or countably infinite (i.e., if it's equivalent to a subset of $\mathbb{N}$).
    - Countable sets can be ***enumerated***, i.e., written as $\{\, a_1, a_2, \ldots, a_n \,\}$ or $\{\, a_1, a_2, a_3, \ldots \}$.
    - Uncountable sets, and thus general infinite sets, cannot!
- Finite sets
  - If A and B are finite, then so are $A \cup B$, $A \cap B$, $A \times B$, and A^B, $\mathscr{P}(A)$, and any $C \subset A$.
    - The same holds for finite combinations of these operations (proof by induction).
  - Finite sets can nevertheless be very very large, e.g., $(10^{10^{10}})$!
- Countable sets
  - Basic theorems:
    - A subset of a countable set is countable.
    - Countable unions and intersections of countable sets are countable.
    - Finite products and finite powers of countable sets are countable.
  - $\mathbb{Q}$ is countable, as is the set of algebraic numbers in $\mathbb{R}$.
- Uncountability
  - Cantor's diagonal argument shows that $\mathbb{R}$, $\{0, 1\}^{\mathbb{N}}$, $\mathscr{P}(\mathbb{N})$, $\mathbb{N} \times \mathbb{N} \times \cdots$, and the Cantor set are all uncountable.
  - Because $\mathbb{R}$ is uncountable and $\mathbb{Q}$ is countable, the set $\mathbb{R} \setminus \mathbb{Q}$ of irrational numbers is uncountable.
  - Consequently, we see that we countable products and countable powers of countable sets need not be countable.
- Every sequence in $\mathbb{R}$ has a monotone subsequence; consequences:
  - Every bounded sequence in $\mathbb{R}$ has a convergent subsequence [Bolzano-Weierstrass].
  - Every Cauchy sequence in $\mathbb{R}$ converges [metric completeness of $\mathbb{R}$].

The Cantor Set and Monotone Functions   Math 323

The Cantor set

For each $n \geq 0$, let $D_n = \{0,1\}^n$, and for each $d = (d_1, d_2, \ldots, d_n) \in D_n$, let $t_d = \displaystyle\sum_{k=1}^{n} \frac{2d_k}{3^k}$.

Then we define ***Cantor's ternary set*** as $C = [0,1] \setminus \displaystyle\bigcup_{n\geq 0} \bigcup_{d\in D_n} \left(t_d + \frac{1}{3^{n+1}}, t_d + \frac{2}{3^{n+1}} \right)$.

- Characterization: C is the set of all numbers in $[0,1]$ having ternary expansions consisting of only 0's and 2's.
- C has zero "length", but nonetheless is uncountable: $C \sim \{0,2\}^{\mathbb{N}} \sim \mathbb{R}$.
- Curious facts: $C + C = [0,2]$ and $C - C = [-1,1]$.

Monotone functions

- Given a set $A \subset \mathbb{R}$, the ***characteristic function*** of A is defined by $\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \text{ or} \\ 0 & \text{if } x \notin A. \end{cases}$
- A monotone function on (a,b) can have at most countably many discontinuities, all of which must be jump discontinuities.
  - Consequence: if $f : [a,b] \to [c,d]$ is monotone and onto, then f must be continuous.
- The ***Cantor-Lebesgue singular function*** can be defined as $f(x) = \displaystyle\sup_{t_d \leq x} \sum_{k=1}^{n} \frac{d_k}{2^k}$ (notation as above).
  - f is continuous and increasing, despite the fact that $f' = 0$ at every point of $[0,1] \setminus C$.
- For any countable set $D \subset \mathbb{R}$, we can construct a function $f : \mathbb{R} \to \mathbb{R}$ that is discontinuous precisely on D.
  - Construction: Let $D = \{ x_1, x_2, \ldots \}$ and choose a positive convergent series $\sum_{n=1}^{\infty} \varepsilon_n$;
    set $f(x) = \sum_{n=1}^{\infty} \varepsilon_n \chi_{[x_n, \infty)}$.

Normed linear spaces

- A ***norm*** on a real linear space X is a function $\|\cdot\| : X \to [0, \infty)$ satisfying three properties:
  - [Positive-definite] $\forall\, x \in X,\ \|x\| = 0 \iff x = 0.$
  - [Homogeneous] $\forall\, x \in X$ and $\alpha \in \mathbb{R},\ \|\alpha x\| = |\alpha|\,\|x\|.$
  - [$\triangle$ inequality] $\forall\, x, y \in X,\ \|x + y\| \leq \|x\| + \|y\|.$
- A ***normed linear space*** is a real linear space X equipped with a norm.
- The norms $\|\cdot\|_p$ on $\mathbb{R}^n$ are defined as follows:
  - For $1 \leq p < \infty$, $\|x\|_p = \left(\sum |x_i|^p\right)^{1/p}$;
    of particular note are the ***one-norm***, $\|x\|_1 = \sum |x_i|$ and the ***Euclidean two-norm***, $\|x\|_2 = \sqrt{\langle x, x\rangle}$.
  - For $p = \infty$, we have the ***sup-norm***, $\|x\|_\infty = \sup |x_i|$ (For $\mathbb{R}^n$, this sup is just a max)
    (this norm actually is, in a quantifiable way, the "limit" of the p-norms as $p \to \infty$).
- Each of the above norms can be extended to the linear space ℓ_p of *real sequences* x for which $\|x\|_p < \infty$.
- Similar norms exists for the set $C[a, b]$ of continuous real-valued functions on $[a, b]$:
  - $\|f\|_1 = \int_a^b |f|$,
  - $\|f\|_2 = \left(\int_a^b |f|^2\right)^{1/2}$,
  - and in general, $\|f\|_p = \left(\int_a^b |f|^p\right)^{1/p}$;
  - $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$ (because f is continuous on $[a, b]$, this sup is just a max).
- Let $1 \leq p, q \leq \infty$ be ***conjugate*** exponents, i.e., suppose that $\frac{1}{p} + \frac{1}{q} = 1$ (taking $\frac{1}{\infty} = 0$);
  conjugate norms satisfy ***Hölder's inequality***: If $x \in \ell_p$ and $y \in \ell_q$, then $\sum |x_i y_i| \leq \|x\|_p\,\|y\|_q$.
  - This is proven using ***Young's inequality***: If $a, b \geq 0$, then $ab \leq \dfrac{a^p}{p} + \dfrac{b^q}{q}$.
  - In the self-conjugate case $p = q = 2$, this is the familiar ***Cauchy-Schwarz inequality***: $|\langle x, y\rangle| \leq \|x\|_2\,\|y\|_2$.
- The triangle inequality for ℓ_p goes by the name ***Minkowski's inequality***: If $x, y \in \ell_p$, then $\|x + y\|_p \leq \|x\|_p + \|y\|_p$.

Metric spaces

- A ***metric*** on a set X is function $d : X \times X \to [0, \infty)$ satisfying three properties:
  - [Positive-definite] $\forall\, x, y \in X,\ d(x, y) = 0 \iff x = y.$
  - [Symmetric] $\forall\, x, y \in X,\ d(x, y) = d(y, x).$
  - [$\triangle$ inequality] $\forall\, x, y, z \in X,\ d(x, y) + d(y, z) \geq d(x, z).$

  We think of the number $d(x, y)$ as the "distance" between x and y.
- A ***metric space*** is a set X equipped with a metric; for example:
  - Any set X can be equipped with the ***discrete metric*** $d(x, y) = \begin{cases} 0 & \text{if } x = y, \text{ or} \\ 1 & \text{if } x \neq y. \end{cases}$
  - The "usual" metric on $\mathbb{R}$ is defined by $d(x, y) = |x - y|$.
  - Any subset A of a metric space X can be given the metric induced by that on X, giving a ***subspace*** of X.
  - Any normed linear space X has a metric given by $d(x, y) = \|x - y\|$.
    - This is the metric we use for any normed linear space unless explicitly noted otherwise.

Suppose that (M, d) is a metric space.

Balls and neighborhoods

Let $x \in M$ and $r > 0$.

- The ***open ball*** of radius r about x is $B_r(x) = \{ y \in M : d(x, y) < r \}$.
  - The critical property of an open ball is that it contains *all* points of M that lie within some distance of x.
  - The vast majority of metric topology is phrased in terms of open balls; a couple of useful variants are below.
- The ***closed ball*** of radius r about x is $\overline{B}_r(x) = \{ y \in M : d(x, y) \leq r \}$.
  - Note that the only difference between a closed ball and an open ball is that the closed ball includes the points at distance exactly r units from the center, while the open ball does not.
- A ***neighborhood*** of x in M is a subset $N \subset M$ for which $\exists\, r > 0$ with $B_r(x) \subset N$.
  - i.e., a neighborhood of x is a set of points that fully contains some open ball about x; we use neighborhoods when we want to discuss sets that contain all points near x but don't want to restrict ourselves just to open balls.

Boundedness and diameter

Suppose that $A \subset M$.

- A is ***bounded*** means $\exists\, x \in M$ and $r > 0$ for which $A \subset B_r(x)$.
- The ***diameter*** of A, $\operatorname{diam} A = \sup \{ d(a, b) : a, b \in A \}$.
- A is bounded $\iff \operatorname{diam} A < \infty$.

Sequences in metric spaces

Let (x_n) be a sequence in M.

Recall that we think of a sequence in M as an infinite, ordered list of elements $(x_1,\ x_2,\ x_3,\ \ldots)$ in M, and that our primary concern with a sequence (x_n) is what happens as we discard more and more of its initial terms, i.e., what happens to the sets $\{ x_n : n \geq N \}$ as we increase N.

- We say that (x_n) ***eventually*** has some property P if $\exists\, N \in \mathbb{N}$ such that $\{ x_n : n \geq N \}$ satisfies P.
  - i.e., if we discard enough initial terms of the sequence, the rest have the property P.
- (x_n) ***converges to*** $x \in M$, written $(x_n) \to x$, means that $(d(x_n, x)) \to 0$.
  - Note that this definition is built from our definition of convergence in $\mathbb{R}$, so it can be unraveled into several logically equivalent forms (see p. 46 for the full discussion):
    - $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N,\ d(x_n, x) < \varepsilon$.
    - $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ such that $\{ x_n : n \geq N \} \subset B_\varepsilon(x)$.
    - $\forall\, \varepsilon > 0$, (x_n) is eventually contained within $B_\varepsilon(x)$.
    - For each neighborhood N of x, (x_n) is eventually contained within N.
  - We say that (x_n) ***converges*** if $\exists\, x \in M$ such that $(x_n) \to x$.
  - We say that (x_n) ***diverges*** if it does not converge (i.e., $\forall\, x \in M,\ (x_n) \not\to x$).
- (x_n) is ***Cauchy*** means that $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ such that $\forall\, n, m \geq N,\ d(x_n, x_m) < \varepsilon$.
  - i.e., as we discard more and more initial terms of the sequence, the rest of the terms squeeze ever closer together; equivalently, that $(\operatorname{diam} \{ x_n : n \geq N \})_N \to 0$.
- Some of the standard results for sequences extend to general metric spaces (see problems 33–39); e.g.:
  - If $(x_n) \to x$ and $(x_n) \to x'$, then $x = x'$. [Limits are unique]
  - If $(x_n) \to x$, then every subsequence $(x_{n_k}) \to x$ as well.
  - (x_n) convergent $\implies (x_n)$ Cauchy $\implies (x_n)$ bounded.
  - A Cauchy sequence with a convergent subsequence converges.
- On the other hand, some results for real sequences do not hold in general metric spaces — these are the theorems that somehow depend extra structure present in $\mathbb{R}$ (see p. 47); e.g.:
  - Cauchy sequences needn't converge. [$\mathbb{R}$ is metrically complete]
  - Bounded sequences needn't have convergent subsequences, [Closed balls in $\mathbb{R}$ are compact]
    nor even Cauchy subsequences. [Closed balls in $\mathbb{R}$ are totally bounded]

Basic Metric Topology, part II — Math 323

Suppose that (M, d) is a metric space.

Open sets
$U \subset M$ is ***open*** [in M] means that $\forall\, x \in U,\ \exists\, \varepsilon > 0$ such that $B_\varepsilon(x) \subset U$.

- General properties of open sets:
  - $\varnothing$ and M are open sets in M.
  - If $\mathscr{U}$ is a collection of open sets, then $\bigcup \mathscr{U}$ is open. [Unions of open sets are open]
  - If $n \in \mathbb{N}$ and $U_1, U_2, \ldots, U_n$ are open sets, then $\bigcap_{k=1}^n U_k$ is open. [*Finite* intersections of open sets are open]
- Examples of open sets:
  · Open balls $B_r(x)$ in any metric space · Open intervals in $\mathbb{R}$ · Any subset of a discrete space
- The ***interior*** of $A \subset M$ is $A^\circ = \bigcup \{\, U : U \text{ is open and } U \subset A \,\}$.
  - Characterization: A° is the largest open set contained in A.
  - Equivalent formulation: $x \in A^\circ \iff \exists\, \varepsilon > 0$ such that $B_\varepsilon(x) \subset A$.
  - The interior operation allows us to *shrink* any set into an open set.
- Every open set in $\mathbb{R}$ is a *countable disjoint* union of open intervals. [Classification of open subsets of $\mathbb{R}$]

Closed sets
$F \subset M$ is ***closed*** [in M] means that $M \setminus F$ is open in M.

- General properties of closed sets (all consequences of DeMorgan's Laws and the corresponding properties of open sets):
  - $\varnothing$ and M are closed sets in M.
  - If $\mathscr{F}$ is a collection of closed sets, then $\bigcap \mathscr{F}$ is closed. [Intersections of closed sets are closed]
  - If $n \in \mathbb{N}$ and $F_1, F_2, \ldots, F_n$ are closed sets, then $\bigcup_{k=1}^n F_k$ is closed. [*Finite* unions of closed sets are closed]
- Examples of closed sets:
  · Every finite set · Intervals in $\mathbb{R}$ whose finite endpoints are closed · Any subset of a discrete space
- The following are equivalent: [The fundamental theorem of closed sets]
  - $M \setminus F$ is open in M. [F is closed in M]
  - $x \in F \iff \forall\, \varepsilon > 0,\ B_\varepsilon(x) \cap F \neq \varnothing$. [$x \in F \iff$ every neighborhood of x intersects F]
  - $\forall\, (x_n) \subset F$, if $(x_n) \to x \in M$, then $x \in F$. [Sequences in F that converge in M must have their limits in F]
- The ***closure*** of $A \subset M$ is $\overline{A} = \bigcap \{\, F : F \text{ is closed and } F \supset A \,\}$.
  - Equivalent formulations: $x \in \overline{A} \iff \forall\, \varepsilon > 0,\ B_\varepsilon(x) \cap A \neq \varnothing \iff \exists\, (a_n) \subset A$ with $(a_n) \to x$.
  - Characterization: $\overline{A}$ is the smallest closed set containing A.
  - The closure operation allows us to *grow* any set into a closed set.

Remarks and related concepts
Let $A \subset M$.

- The concepts "open" and "closed"...
  - are *not* mutually exclusive; a set could be both (e.g., $\varnothing$) or neither (e.g., $[0, 1) \subset \mathbb{R}$).
  - depend on the containing space, i.e., are not intrinsic to the set. If $A \subset N \subset M$, then:
    · A is open in $N \iff \exists$ an open set $U \subset M$ with $A = U \cap N$; and
    · A is closed in $N \iff \exists$ a closed set $F \subset M$ with $A = F \cap N$.

    e.g., the interval $[0, 1)$ is both open and closed in $N = [0, 1)$, but neither open nor closed in $M = \mathbb{R}$.
- A is ***dense*** in M if $\overline{A} = M$.
  - Equivalent formulation: A is dense $\iff \forall\, x \in M$ and $r > 0,\ B_r(x) \cap A \neq \varnothing$.
  - A metric space M is called ***separable*** if it contains a *countable* dense subset
    · e.g., $\mathbb{R}$ is separable, because $\mathbb{Q} \subset \mathbb{R}$ is dense in $\mathbb{R}$.
- The ***boundary*** of A is $\partial A = \overline{A} \cap \overline{M \setminus A}$.
  - For any $A \subset M$, M decomposes into a disjoint union $A^\circ \cup \partial A \cup (M \setminus A)^\circ$.
- $x \in M$ is a ***limit point*** of A means $\forall\, \varepsilon > 0,\ (B_\varepsilon(x) \setminus \{x\}) \cap A \neq \varnothing$.
  - The concept of limit point is the worst idea in all of metric topology, and it should be avoided whenever possible.

Suppose that (M, d) and (N, ρ) are metric spaces.

Continuity of functions

Suppose that $f : M \to N$.

- f is ***continuous at a point*** $x \in M$ means $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that $\forall\, y \in M,\ d(x, y) < \delta \implies \rho(f(x), f(y)) < \varepsilon$.
  - Equivalent formulations: · $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that $f[B_\delta(x)] \subset B_\varepsilon(f(x))$; or
    · $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that $B_\delta(x) \subset f^{-1}[B_\varepsilon(f(x))]$.
  - In words: for each neighborhood of $f(x) \in N$, there is some neighborhood of $x \in M$ mapped entirely into it by f. or, f^{-1} maps each neighborhood of $f(x)$ to a neighborhood of x.
- f is ***continuous*** [on M] means $\forall\, x \in M,\ f$ is continuous at x.
  - Equivalent formulations: [The fundamental theorem of continuity]
    - $\forall\, (x_n) \subset M$ and $x \in M$, $(x_n) \to x \implies (f(x_n)) \to f(x)$.
    - $\forall$ closed $E \subset N,\ f^{-1}[E]$ is closed in M.
    - $\forall$ open $V \subset N,\ f^{-1}[V]$ is open in M.
- f is a ***Lipschitz*** function means that $\exists\, K < \infty$ such that $\forall\, x, y \in M,\ \rho(f(x), f(y)) \le K \cdot d(x, y)$.
  - i.e., a Lipschitz function is one that doesn't stretch its domain by more than some fixed factor.
- Important examples:
  - Constant functions and the identity map are continuous.
  - All Lipschitz functions are continuous.
  - All real polynomials are continuous on $\mathbb{R}$.
  - The function $\chi_{\mathbb{Q}}$ is discontinuous at every point of $\mathbb{R}$.
  - The function $\sup\limits_{n\in\mathbb{N}} \left[\frac{1}{n} \cdot \chi_{\frac{1}{n}\mathbb{Z}}\right]$ is discontinuous at each point of $\mathbb{Q}$ and continuous at each point of $\mathbb{R} \setminus \mathbb{Q}$.
- Distance functions
  - The metric $d : M \times M \to [0, \infty)$ is a continuous function on $M \times M$.
  - Given a nonempty subset $A \subset M$, we define $d_A(x) = \inf\{\, d(x, a) : a \in A \,\}$.
    - This function $d_A : M \to [0, \infty)$ is continuous on M, and thus preimages of open/closed sets are open/closed.
    - In particular, $\overline{A} = d_A^{-1}[\{0\}]$—i.e., $x \in \overline{A} \iff d_A(x) = 0$.
- A ***homeomorphism*** is a bijective function $f : M \to N$ such that both f and f^{-1} are continuous.
  Equivalent conditions: · $(x_n) \to x \iff (f(x_n)) \to f(x)$
  · $G \subset M$ is open $\iff f[G] \subset N$ is open
  · $E \subset M$ is closed $\iff f[E] \subset N$ is closed
- An ***isometry*** is a bijective function $F : M \to N$ such that $\forall\, x, y \in M,\ \rho(f(x), f(y)) = d(x, y)$.
  - Every isometry is bicontinuous, and thus is a homeomorphism.

Building continuous functions

- Compositions: Suppose that $L \xrightarrow{f} M \xrightarrow{f} N$.
  - If f is continuous at $x \in L$ and g is continuous at $f(x) \in M$, then $g \circ f : L \to N$ is continuous at x.
  - Consequently, if f is continous and g is continuous, then $g \circ f$ is continuous.
- Stackings: Suppose that $f, g : M \to N$ are continuous, and define $f \,\square\, g : M \to N \times N$ by $x \mapsto (f(x), g(x))$.
  - If f and g are continuous at $x \in M$, then $f \,\square\, g$ is continuous at x.
  - Consequently, if f and g are continuous, then $f \,\square\, g$ is continuous.
- Basic operations in $\mathbb{R}$: The following functions are continuous:
  · $+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ · $- : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ · $\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ · $\div : \mathbb{R} \times (\mathbb{R} \setminus \{0\}) \to \mathbb{R}$
  · $\max : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ · $\min : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
  - Thus, if $f, g : M \to \mathbb{R}$ are continuous at $x \in M$, then so are $f \pm g,\ f \cdot g,\ \max(f, g),$ and [if $g(x) \neq 0$] $f \div g$.
  - Consequently, if $f, g : M \to \mathbb{R}$ are continuous, then so are $f \pm g,\ f \cdot g,\ \max(f, g),$ and [if g is nonzero] $f \div g$.
- The set $C(M)$ of all continuous functions $f : M \to \mathbb{R}$ forms both an algebra (and thus a vector space) and a lattice.
  - i.e., we can add, subtract, multiply, and take $\min$'s and $\max$'s in $C(M)$.

Basic Metric Topology, part III — Math 323

Let (M, d) be a metric space.

Connectedness

There are several equivalent formulations of what it means for M to be ***connected***:

- If $A, B \subset M$ are disjoint and open with $M = A \cup B$, then $A = \varnothing$ or $B = \varnothing$.
- $\varnothing$ and M are the only ***clopen*** subsets of M.
- $\nexists$ a continuous surjection $f : M \to \{0, 1\}$.

- $A \subset \mathbb{R}$ is connected $\iff$ A is an interval. [Connectedness in $\mathbb{R}$]
- If M is connected and $f : M \to N$ is continuous, then $f[M]$ is connected.
- If M, N are connected spaces, then $M \times N$ is connected.

Total boundedness

A subset $A \subset M$ is called ***totally bounded*** if $\forall\, \varepsilon > 0,\ \exists\, x_1, x_2, \ldots, x_n \in M$ such that $A \subset \bigcup_{k=1}^{n} B_\varepsilon(x_k)$.

- Equivalent formulation: A is totally bounded $\iff$ every sequence in A has a Cauchy subsequence.
  - Corollary: Every bounded infinite subset of $\mathbb{R}$ has a limit point in $\mathbb{R}$. [Bolzano-Weierstrass theorem]
- Examples:
  - All finite sets are totally bounded.
  - For $A \subset \mathbb{R}^n$, A is bounded $\iff$ A is totally bounded.
  - Totally bounded $\implies$ bounded, but not vice versa (e.g., in ℓ_2).

Completeness

M is ***complete*** means that every Cauchy sequence in M converges to a point in M; equivalently:

- If $F_1 \supset F_2 \supset F_3 \supset \ldots$ is a decreasing sequence of nonempty closed subsets of M such that $(\operatorname{diam} F_n) \to 0$, then $\bigcap_{n \in \mathbb{N}} F_n \neq \varnothing$.
- Every infinite, totally bounded subset of M has a limit point in M.

- Examples of complete spaces: $\mathbb{R},\ \mathbb{R}^n,\ \ell_0,\ \ell_1,\ \ell_2,\ \ell_\infty,\ C[a, b]$, any discrete space.
- Suppose that M is complete. Then $A \subset M$ is complete $\iff$ A is closed in M.
- A ***Banach space*** is a *complete* normed vector space.
  - A normed linear space X is complete $\iff$ $[\forall\, (x_n) \subset X,\ \sum_{n=1}^{\infty} x_n$ converges $\iff$ $\sum_{n=1}^{\infty} \|x_n\|$ converges$]$.
    [i.e., the convergence of a series in X depends solely on the convergence of its series of norms.]

Compactness

M is ***compact*** means that M is complete and totally bounded; equivalently:

- Every sequence in M has a subsequence that converges to a point in M.
- For each collection $\mathcal{G}$ of open subsets of M with $\bigcup \mathcal{G} = M$, $\exists\, G_1, G_2, \ldots, G_n \in \mathcal{G}$ such that $\bigcup_{k=1}^{n} G_k = M$.
  [Every ***open cover*** of M has a finite ***subcover***]
- If $\mathcal{F}$ is a collection of closed subsets of M such that every finite subcollection $\mathcal{F}' \subset \mathcal{F}$ has $\bigcap \mathcal{F}' \neq \varnothing$, then $\bigcap \mathcal{F} \neq \varnothing$. [The "finite intersection property"]

- Compactness of subspaces: Suppose that $A \subset M$.
  - If A is compact, then A is closed in M. [Compact $\implies$ closed]
  - If M is compact and A is closed, then A is compact. [Closed subspaces of compact spaces are compact]
- If M is compact and $f : M \to N$ is continuous, then $f[M]$ is compact.
  Corollary: If M is nonempty, compact, and connected and $f : M \to \mathbb{R}$ is continuous, then $\exists\, c, d \in \mathbb{R}$ such that $f[M] = [c, d]$.
  [i.e., such a function must be bounded and must attain its supremum and infimum; the image is an interval by connectedness]

Calculus results

- Suppose that $f : [a,b] \to \mathbb{R}$ is continuous. Then:
  - [The Intermediate Value Theorem] f assumes all values between $f(a)$ and $f(b)$.
  - [The Extreme Value Theorem] f has a maximum value and a minimum value on $[a,b]$.
- Suppose that $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b). Then:
  - [Fermat's Theorem] If f has a local extremum at $c \in (a,b)$, then $f'(c) = 0$.
    - Thus, the local extrema of a continuous function $f : [a,b] \to \mathbb{R}$ can only occur at:
      (i) the endpoints a and b, or
      (ii) a ***critical point*** c for f, i.e., a point $c \in (a,b)$ at which f' is zero or undefined.
  - [Rolle's Theorem] If $f(a) = f(b)$, then $\exists\, c \in (a,b)$ such that $f'(c) = 0$.
  - [The Mean Value Theorem] $\exists\, c \in (a,b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.
- [The Generalized Mean Value Theorem] If $f, g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), then $\exists\, c \in (a,b)$ such that $f'(c)\,[g(b) - g(a)] = g'(c)\,[f(b) - f(a)]$.
- [l'Hôpital's Rule] If either $f(t), g(t) \to 0$ or $g(t) \to \infty$, and if $\dfrac{f'(t)}{g'(t)} \to \alpha$, then $\dfrac{f(t)}{g(t)} \to \alpha$ as well.
  [Where the limits are as $t \to$ any of $\{\, a^+,\ a^-,\ a^\pm,\ +\infty,\ -\infty \,\}$, and α can be any of $\{\, L, +\infty, -\infty \,\}$.]

Fixed points

Let (M,d) be a metric space, and let $f : M \to M$.

- A ***fixed point*** of f is a point $x \in M$ with $f(x) = x$.
- $f : M \to M$ is a ***contraction mapping*** if $\exists\, 0 \le \alpha < 1$ such that $\forall\, x, y \in M,\ d(f(x), f(y)) \le \alpha \cdot d(x,y)$.
  - Contraction mappings are continuous (e.g., because they are Lipschitz).
  - Note the strict inequality $\alpha < 1$, and that it is not enough merely that $\forall\, x, y \in M,\ d(f(x), f(y)) \le d(x,y)$.
- The Contraction Mapping Theorem: If M is complete and $f : M \to M$ is a contraction mapping, then f has a unique fixed point $x_0 \in M$, and for any $x \in M$, the sequence of iterates $(f^n(x)) \to x_0$.
  - This is a very strong tool, both for proving existence of a solution to an equation and for estimating that solution.
  - Some applications:
    - Estimating values in $\mathbb{R}$, e.g., n^{th} roots and other difficult values.
    - Existence and uniqueness of solutions to certain differential equations.
    - Existence of fractal sets in $\mathbb{R}^n$.

Uniform continuity

Let (M,d) and (N,ρ) be metric spaces.

Recall that $f : M \to N$ is continuous means $\forall\, x \in M,\ \forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that $f[B_\delta(x)] \subset B_\varepsilon(f(x))$; because of the order of quantifiers, $x \in M$ and $\varepsilon > 0$ must both be specified *before* the $\delta > 0$ is demanded — thus, the resulting δ could depend not only on ε, but on x as well. Uniform continuity demands that the same $\delta > 0$ work for all $x \in M$ simultaneously:

f is ***uniformly continuous*** means $\forall\, \varepsilon > 0,\ \exists\, \delta > 0$ such that $\forall\, x \in M,\ f[B_\delta(x)] \subset B_\varepsilon(f(x))$.

- Consequences of uniform continuity: If f is uniformly continuous, then…
  - f is continuous.
  - If $(x_n) \subset M$ is a Cauchy sequence, then $(f(x_n)) \subset N$ is also a Cauchy sequence.
  - If $A \subset M$ is totally bounded, then $f[A] \subset N$ is totally bounded.
    - Note that the above two conclusions do not necessarily the case for functions that are merely continuous!
- If $f : M \to N$ is continuous and M is compact, then f is uniformly continuous.
- If D is dense in M, N is complete, and $f : D \to N$ is uniformly continuous, then f extends uniquely to a uniformly continuous function $\tilde{f} : M \to N$.
  - Consequently, a continuous function $f : \mathbb{R} \to \mathbb{R}$ is determined by its values on $\mathbb{Q}$ (why?—be careful!).

Discontinuity and Category Math 323

Discontinuity

Let $f : \mathbb{R} \to \mathbb{R}$.

- Let $D(f) = \{\, x \in \mathbb{R} \,:\, f \text{ is discontinuous at } x \,\}$.
  We've seen that $D(f)$ could be:
  - $\varnothing$ (if f is continuous)
  - Any countable set in $\mathbb{R}$ (e.g., for a monotone function)
  - $\mathbb{Q}$, or even $\mathbb{R}$ (examples given in discussion of continuity)
- For any interval $I \subset \mathbb{R}$, the ***oscillation of** f **on** I* is $\omega(f; I) = \sup\{\, |f(x) - f(y)| \,:\, x, y \in I \,\}$.
  - $\omega(f; I)$ measures how widely the values of f vary on I.
  - Basic properties:
    - $0 \le \omega(f; I) \le 2\,\|f\|_\infty$.
    - $I \subset J \implies \omega(f; I) \le \omega(f; J)$.
- The ***oscillation of** f **at*** $a \in \mathbb{R}$ has several equivalent formulations:
  $$\omega_f(a) = \inf_{\text{open } I \ni a} \omega(f; I) = \lim_{h \to 0^+} \omega(f; (a-h, a+h)) = \lim_{h \to 0^+} \operatorname{diam} f[B_h(a)].$$
  - $\omega_f(a) \ge 0$ measures how widely the values of f vary "near" a.
  - f is continuous at $a \iff \omega_f(a) = 0$. [i.e., $a \in D(f) \iff \omega_f(a) > 0$.]
- Structure theorem for $D(f)$: If $f : \mathbb{R} \to \mathbb{R}$, then $D(f)$ is a countable union of closed sets in $\mathbb{R}$ (i.e., an F_σ in $\mathbb{R}$).
  - In fact, every F_σ in $\mathbb{R}$ is the set of discontinuities of some function $f : \mathbb{R} \to \mathbb{R}$.

Category

Let (M, d) be a metric space.

Basic definitions:

- A F_σ in M is a countable union of closed sets in M.
  - e.g., $\mathbb{Q}$ (or any countable set) in $\mathbb{R}$, or any open or closed set
- A G_δ in M is a countable intersection of open sets in M.
  - e.g., $\mathbb{R} \setminus \mathbb{Q}$ (or the complement of any countable set) in $\mathbb{R}$, or any open or closed set
- $E \subset M$ is ***nowhere dense*** means $\overline{E}^\circ = \varnothing$. [equivalently, $M \setminus \overline{E}$ is open and dense in M]
- $A \subset M$ is of the ***first category*** in M if it is a countable union of nowhere dense sets in M.
  - e.g., $\mathbb{Q}$ (or any countable set) in $\mathbb{R}$ is of the first category in $\mathbb{R}$.
- $B \subset M$ is of the ***second category*** in M if it is not of the first category.
  - i.e., B cannot be written as a countable union of nowhere dense sets in M.

The Baire Category Theorem: If M is a complete metric space, then M is of the second category in itself.
[Equivalently, if (G_n) is a sequence of dense open sets in a complete metric space M, then $\bigcap_{n \in \mathbb{N}} G_n$ is dense in M.]

Consequences:

- A dense G_δ in $\mathbb{R}$ must be uncountable (consider removing one point from each open set in the intersection).
- $\mathbb{Q}$ is not a G_δ in $\mathbb{R}$ (because it's dense and countable), and thus $\mathbb{R} \setminus \mathbb{Q}$ is not $D(f)$ for any $f : \mathbb{R} \to \mathbb{R}$.
- If we express $\mathbb{R}$ (or even $\mathbb{R} \setminus \mathbb{Q}$) as an F_σ,
  then at least one of the closed sets must have a nonempty interior.

Pointwise and Uniform Convergence

Let (M,d) and (N,ρ) be metric spaces, and let (f_n) be a sequence of functions from $M \to N$.

Pointwise convergence

We say that (f_n) ***converges pointwise*** to f if $\forall\, x \in M,\ \forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ such that $\forall\, n \geq N,\ \rho(f_n(x), f(x)) < \varepsilon$, i.e., $\forall\, x \in M,\ (f_n(x)) \to f(x)$.

- Note that in this definition, N can depend on both x and ε, so the rate of convergence can vary from point to point.
- In general, pointwise convergence need not respect…
  - Limits and continuity: Let $f_n(x) = \chi_{[0,1]} \cdot x^n$, and let $f = \chi_{\{1\}}$; then $(f_n) \to f$ pointwise on $[0,1]$.
    Each f_n is continuous on $[0,1]$ yet f is not; also, $\lim\limits_{n\to\infty}\left[\lim\limits_{x\to 1^-} f_n(x)\right] \neq \lim\limits_{x\to 1^-}\left[\lim\limits_{n\to\infty} f_n(x)\right]$.
  - Integrals: Let $g_n(x) = \chi_{[0,\frac{2}{n}]} \cdot 2n(1 - |x - \frac{1}{n}|)$, and let $g(x) = 0$; then $(g_n) \to g$ pointwise on $[0,1]$.
    $(\int_0^1 g_n) \not\to \int_0^1 g$, i.e., $\lim\limits_{n\to\infty}\left[\int_0^1 g_n\right] \neq \int_0^1 \left[\lim\limits_{n\to\infty} g_n\right]$.
  - Derivatives: Let $h_n(x) = x^{n+1}/(n+1)$, and let $h(x) = 0$; then $(h_n) \to h$ pointwise on $[0,1]$.
    $(h_n'(1)) \not\to (h'(1))$, i.e., $\lim\limits_{n\to\infty} h_n' \neq \left[\lim\limits_{n\to\infty} h_n\right]'$.

  In summary, we cannot interchange pointwise limits and limits, integrals, or derivatives!

Uniform convergence

We say that (f_n) ***converges uniformly*** to f if $\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ such that $\forall\, x \in M,\ \forall\, n \geq N,\ \rho(f_n(x), f(x)) < \varepsilon$, i.e., $(\|f_n - f\|_\infty) \to 0$.

- Notation: We write $(f_n) \rightrightarrows f$ to indicate uniform convergence.
  $(f_n) \to f$ indicates pointwise convergence unless otherwise specified [e.g., "$(f_n) \to f$ in $C[0,1]$" means $(f_n) \rightrightarrows f$].
- In this definition, for each $\varepsilon > 0$, the same N must work simultaneously for all x.
- Uniform convergence $\implies$ pointwise convergence.
- Uniform convergence and…
  - Continuity: If each f_n is continuous and $(f_n) \rightrightarrows f$, then f is continuous.
    - To rephrase the conclusion: if $(x_m) \to x$ in M, then $\lim\limits_{n\to\infty}\lim\limits_{m\to\infty} f_n(x_m) = \lim\limits_{m\to\infty}\lim\limits_{n\to\infty} f_n(x_m)$.
    - Consequence: the metric space $C[a,b]$ is "closed" under uniform limits.
  - Integrals: If each $f_n : [a,b] \to \mathbb{R}$ is continuous and if $(f_n) \rightrightarrows f$ on $[a,b]$, then $\left(\int_a^b f_n\right) \to \int_a^b f$.
  - Derivatives: If (i) Each $f_n : [a,b] \to \mathbb{R}$ has a continuous derivative on $[a,b]$;
    (ii) $(f_n') \rightrightarrows g$ on on $[a,b]$; and
    (iii) $\exists\, x_0 \in [a,b]$ for which $(f_n(x_0))$ converges,
    then $\exists$ a differentiable function $f : [a,b] \to \mathbb{R}$ and with $f' = g$, i.e., $(f_n') \rightrightarrows f'$.

Applications of the Weierstrass M-Test Math 323

The space of bounded functions

Let X be a set.

- We define $B(X)$ to be the normed algebra of all bounded functions $f : X \to \mathbb{R}$, equipped with the sup-norm $\|\cdot\|_\infty$.
  - $B(X)$ is complete, i.e., every Cauchy sequence (f_n) in $B(X)$ converges [uniformly] to some function $f \in B(X)$.
  - Note that such a Cauchy sequence (f_n) is bounded and $(\|f_n\|_\infty) \to \|f\|_\infty$.
- If X happens to be a metric space, we define $C_b(X) = C(X) \cap B(X)$.
  i.e., $C_b(X)$ is the set of all continuous, bounded real-valued functions on X.
  - $C_b(X) \subset B(X)$ is closed, and thus complete, i.e., every Cauchy sequence in $C_b(X)$ converges in $C_b(X)$.

The Weierstrass M-Test

Let X be a set.

The Weierstrass M-Test: (for uniform convergence) If (g_n) is a sequence in $B(X)$ with $\sum_{n=1}^\infty \|g_n\|_\infty < \infty$, then $\sum_{n=1}^\infty g_n$ converges uniformly on X. [and $\|\sum_{n=1}^\infty g_n\|_\infty \le \sum_{n=1}^\infty \|g_n\|_\infty$]

The Weierstrass M-Test yields a number of interesting results:

- Power series: If the power series $\sum_{n=0}^\infty a_n x^n$ converges for some $x \neq 0$,
  then it converges, uniformly and absolutely, on every interval $[-R, R]$ with $0 < R < |x_0|$.
  - Thus, such a power series defines a continuous function f on the interval $I = (-|x_0|, |x_0|)$ such that:
    - $\forall\, x \in I,\ f'(x) = \displaystyle\sum_{n=1}^\infty n a_n x^{n-1}$, and
    - $\forall\, [a,b] \subset I,\ \displaystyle\int_a^b f = \left[\sum_{n=0}^\infty \frac{a_n x^{n+1}}{n+1}\right]_a^b$.
- A space-filling curve: $\exists$ a continuous surjection $\gamma : [0,1] \to [0,1] \times [0,1]$.
  - If we're careful, we can even arrange that γ maps the Cantor set onto $[0,1] \times [0,1]$.
- Discontinuous functions: Given any F_σ in $\mathbb{R}$, $\exists$ a function $f : \mathbb{R} \to \mathbb{R}$ for which $D(f)$ is that set.
- A continuous, nowhere-differentiable function $f : \mathbb{R} \to \mathbb{R}$:
  The function given by $f(x) = \sum_{n=0}^\infty 2^{-n}\, d(x, 2^{-n}\mathbb{Z})$ is continuous on $\mathbb{R}$ but not differentiable at any point of $\mathbb{R}$.

C[*a*,*b*]: Approximation and separability

The space $C[0,1]$, and thus $C[a,b]$, is separable (under $\|\cdot\|_\infty$); this can be shown in several ways, generally in two steps:

- Given $f \in C[0,1]$ and $\varepsilon > 0$, approximate f to within $\varepsilon/2$ by either…
  - A polygonal (piecewise-linear) function:
    - Take nodes by sampling f at the points of a partition of $[0,1]$ with spacing $\delta > 0$ given by uniform continuity.
  - Bernstein polynomials: $(B_n(f))(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \cdot \binom{n}{k} x^k (1-x)^{n-k}$:
    - $(B_n(f)) \rightrightarrows f$ in $C[0,1]$.
    - Consequence: If $f \in C[a,b]$ and $\forall\, n \geq 0$, $\int_a^b f(x) \cdot x^n \, dx = 0$, then $f = 0$.
- Replace that function's coefficients by rational numbers so that the result is within $\varepsilon/2$ of the initial approximation; by the triangle inequality, the new approximation is within ε of the original function f.

Net effect: the [countable] set of such "rationalized" approximating functions is dense in $C[0,1]$.

Continuity and category

Continuous functions are "rare" among bounded functions, as are differentiable functions among continuous functions:

- $\{ f \ : \ \exists\, x \in [a,b]$ at which f is differentiable $\}$ is of first category in $C[a,b]$.
- Similarly, $\{ f \ : \ \exists\, x \in [a,b]$ at which f is continuous $\}$ is of first category in $L_\infty[a,b]$. [Not proven here.]

Functions of Bounded Variation

The oscillation of f at a ($\omega_f(a)$) measures how widely f scatters its values near a; however, it does not take into account the actual behavior of the function from point to point, only the set of values that result—e.g., consider the oscillations of $f(x) = \chi_{\mathbb{Q}}$ and $g(x) = \chi_{[0,\infty]}$ at 0. A more refined sense of a function's behavior is given by its ***variation***:

$$V_a^b(f) \stackrel{\text{def}}{=} \sup \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \ : \ a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \right\}.$$

- Examples:
  - If f is monotone on $[a,b]$, then $V_a^b(f) = |f(b) - f(a)|$.
  - If f is Lipschitz on $[a,b]$ with Lipschitz constant K, then $V_a^b f \leq K|b-a|$.
- We define $BV[a,b] = \{ f : V_a^b f < \infty \}$, the functions of ***bounded variation*** on $[a,b]$.
  - From above, [piecewise-]monotone functions and Lipschitz functions are of bounded variation.
  - Bounded variation vs. continuity:
    - $C[a,b] \not\subset BV[a,b]$—e.g., consider $f(x) = x \sin \frac{1}{x}$ on $[0,1]$.
    - $BV[a,b] \not\subset C[a,b]$—e.g., consider $f(x) = \chi_{[0,\infty)}$ on $[-1,1]$.
  - Bounded variation vs. monotonicity: as noted above, every monotone function on $[a,b]$ is in $BV[a,b]$. The converse does not hold—but every function of bounded variation is a *difference* of monotone functions:
    - If $f \in BV[a,b]$, then $\exists$ increasing functions g and h with $f = g - h$. [Set $g(x) = V_a^x f$ and $h = f - g$.] Note that if f is continuous, then g and h may be chosen so as well.
- Variation and the ∞-norm:
  - If $f \in BV[a,b]$, then $\|f\|_\infty \leq |f(a)| + V_a^b f$ (and, consequently, $BV[a,b] \subset L_\infty[a,b]$).
  - In fact, under this ***BV-norm*** $\|f\|_{BV} \stackrel{\text{def}}{=} |f(a)| + V_a^b f$, the space $BV[a,b]$ forms a *complete* metric space.
- Basic properties of V_a^b (all proven via the definition):
  - $V_a^b f = 0$ if, and only if, f is constant.
  - Absolute sub-linearity: $V_a^b(cf) = |c| V_a^b f$, and $V_a^b(f+g) \leq V_a^b f + V_a^b g$.
  - Product rule: $V_a^b(fg) \leq \|f\|_\infty V_a^b g + \|g\|_\infty V_a^b f$.
  - Absolute values: $V_a^b|f| \leq V_a^b f$.
  - Equality under splitting the interval: if $a \leq c \leq b$, then $V_a^b f = V_a^c f + V_c^b f$.

Ingredients and definition

The Riemann-Stieltjes integral extends the classical Riemann integral by allowing us to redefine our "measure" on the domain of integration (allowing us to simultaneously treat both discrete and continuous phenomena—or even mixtures of the two!).

- Ingredients:
  - an interval $[a, b]$ (the interval of integration)
  - a bounded function f on $[a, b]$ (the integrand)
  - an increasing function α on $[a, b]$ (which "measures" intervals $[x_{k-1}, x_k]$ via $\alpha(x_k) - \alpha(x_{k-1})$)
    - Note that we may lift this restriction that α be monotone;
      in this case, we require it instead to be of bounded variation, i.e., a *difference* of increasing functions.
- Given these three ingredients, we define $\int_a^b f\,d\alpha$ as follows:
  - For each ***partition*** $\mathcal{P} = \{\, a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b \,\}$ of $[a, b]$:
    - Let $\Delta\alpha_k = \alpha(x_k) - \alpha(x_{k-1})$ (generalizing the usual $\Delta x_k = x_k - x_{k-1}$ for the Riemann integral).
    - Set $m_k = \inf_{x\in[x_{k-1},x_k]} f(x)$ and $M_k = \sup_{x\in[x_{k-1},x_k]} f(x)$.
    - Define $L(f, \mathcal{P}) = \sum_{k=1}^n m_k \Delta\alpha_k$ and $U(f, \mathcal{P}) = \sum_{k=1}^n M_k \Delta\alpha_k$.
  - We then define the upper and lower integrals as $\underline{\int_a^b} f\,d\alpha = \sup_{\mathcal{P}} L(f, \mathcal{P})$ and $\overline{\int_a^b} f\,d\alpha = \inf_{\mathcal{P}} U(f, \mathcal{P})$.
  - If $\underline{\int_a^b} f\,d\alpha = \overline{\int_a^b} f\,d\alpha$, we write that value as $\int_a^b f\,d\alpha$ and say that $f \in \mathcal{R}_\alpha[a, b]$, i.e., f is ***α-integrable*** on $[a, b]$.

Basic properties

- Interpretation: Suppose that $f \in C[a, b]$.
  - If $\alpha(x) = x$, then $\int_a^b f\,d\alpha = \int_a^b f$.
    - i.e., the Riemann integral is a particular case of the Riemann-Stieltjes integral.
  - If $\alpha(x) = kx$, then $\int_a^b f\,d\alpha = k\int_a^b f$.
    - i.e., when α is differentiable, α' determines how much we "stretch" the domain $[a, b]$ at each point.
  - If $\alpha(x) = k \cdot \chi_{([x_0,\infty)}$ and $x_0 \in (a, b)$, then $\int_a^b f\,d\alpha = k \cdot f(x_0)$.
    - i.e., at discontinuities of α, the RS integral measures the jump in α times the value of f at the jump.
- Properties: Suppose that α is increasing on $[a, b]$; $f, g \in \mathcal{R}_\alpha[a, b]$; and $c \in \mathbb{R}$.
  - Linearity: $\int_a^b cf\,d\alpha = c\int_a^b f\,d\alpha$ and $\int_a^b (f + g)\,d\alpha = \int_a^b f\,d\alpha + \int_a^b g\,d\alpha$.
  - Monotonocity: If $f \le g$ on $[a, b]$, then $\int_a^b f\,d\alpha \le \int_a^b g\,d\alpha$.
  - Absolutely bounded: $|f| \in \mathcal{R}_\alpha[a, b]$ and $|\int_a^b f\,d\alpha| \le \int_a^b |f|\,d\alpha \le \|f\|_\infty\,[\alpha(b) - \alpha(a)]$.
  - Cauchy-Schwarz: $|\int_a^b fg\,d\alpha| \le (\int_a^b f^2\,d\alpha)^{1/2}(\int_a^b g^2\,d\alpha)^{1/2}$.
- The Riemann-Stieltjes integral is also linear in α:
  - If $f \in \mathcal{R}_\alpha[a, b]$ and $c \in \mathbb{R}$, then $f \in \mathbb{R}_{c\alpha}[a, b]$ and $\int_a^b f\,d(c\alpha) = c\int_a^b f\,d\alpha$.
  - If $f \in \mathcal{R}_\alpha[a, b] \cap \mathcal{R}_\beta[a, b]$, then $f \in \mathbb{R}_{\alpha+\beta}[a, b]$ and $\int_a^b f\,d(\alpha + \beta) = \int_a^b f\,d\alpha + \int_a^b f\,d\beta$.
- Integrability conditions:
  - Riemann's condition: Suppose that α is increasing on $[a, b]$ and $f : [a, b] \to \mathbb{R}$.
    Then $f \in \mathcal{R}_\alpha[a, b] \iff \forall\,\varepsilon > 0, \exists\,\mathcal{P}$ for which $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$.
  - If α is monotone on $[a, b]$ and f is continuous on $[a, b]$, then $f \in \mathcal{R}_\alpha[a, b]$ (i.e., $C[a, b] \subset \mathcal{R}_\alpha[a, b]$).
  - If f is monotone on $[a, b]$ and α is continuous on $[a, b]$, then $f \in \mathcal{R}_\alpha[a, b]$.
    - If we allow α to be non-monotone, we can replace "monotone" with "of bounded variation" above.
  - Symmetry: $f \in \mathcal{R}_\alpha[a, b]$ if, and only if, $\alpha \in \mathcal{R}_f[a, b]$ (proven via integration by parts).

Integral theorems

- Variation: If $f' \in \mathcal{R}[a,b]$, then $f \in BV[a,b]$ and $V_a^b f = \int_a^b |f'(t)|\, dt$.
- Uniform convergence: If (f_n) is a sequence in $\mathcal{R}_\alpha[a,b]$ and $(f_n) \rightrightarrows f$, then $f \in \mathcal{R}_\alpha[a,b]$ and $(\int_a^b f_n\, d\alpha) \to \int_a^b f\, d\alpha$.
- Integration by parts: $f \in \mathcal{R}_\alpha[a,b]$ if, and only if, $\alpha \in \mathcal{R}_f[a,b]$, and in either case, $\int_a^b f\, d\alpha + \int_a^b \alpha\, df = f(b)\alpha(b) - f(a)\alpha(a)$.
- Substitution: Suppose that $\alpha' \in \mathcal{R}[a,b]$ and $\|\alpha'\|_\infty, \|f\|_\infty < \infty$.
  Then $f \in \mathcal{R}_\alpha[a,b]$ if, and only if, $f\alpha' \in \mathcal{R}[a,b]$, and in either case, $\int_a^b f\, d\alpha = \int_a^b f(x)\alpha'(x)\, dx$.
  - Corollary (The Second Fundamental Theorem of Calculus):
    If f is differentiable, $f' \in \mathcal{R}[a,b]$, and $\|f'\|_\infty < \infty$, then $\int_a^b f'(x)\, dx = f(b) - f(a)$.
  - Note that this result also tells us how to *compute* $\int_a^b f\, d\alpha$ when α' is nice; as a counterpoint, we have a result about integrating with respect to pure jump functions:
    - For $x_1, x_2, \ldots \in (a,b)$ and $c_1, c_2, \ldots \geq 0$, set $\alpha(x) = \sum_{n=1}^\infty c_k \cdot \chi_{[x_k,\infty)}$.
      Then for each $f \in C[a,b]$, we have $\int_a^b f\, d\alpha = \sum_{k=1}^\infty c_k f(x_k)$.
- The First Fundamental Theorem of Calculus:
  Suppose that α is increasing on $[a,b]$ and $f \in \mathcal{R}_\alpha[a,b]$, and define $F(x) = \int_a^x f\, d\alpha$ for $x \in [a,b]$. Then:
  - $F \in BV[a,b]$,
  - F is continuous everywhere that α is continuous, and
  - F is differentiable at each point where both α is differentiable and f is continuous;
    at these points, $F'(x) = f(x)\alpha'(x)$.

The Riesz Representation Theorem

As we've already seen, the operation of Riemann integration, $\int_a^b : C[a,b] \to \mathbb{R}$ is linear and continuous; in fact, for any function $\varphi \in C[a,b]$, the map $f \mapsto \int_a^b f\varphi$ is also linear and continuous. Of course, not every such linear, continuous function arises in this way (for example, the function $f \mapsto f(a)$, which, unlike any Riemann integral, depends only on the value of f at a). The natural question, then, is whether there is some construction that exhibits the set of *all* continuous linear functions from $C[a,b] \to \mathbb{R}$. The *Riemann-Stieltjes* integral provides just the result we're seeking:

- **The Riesz Representation Theorem** (for the Riemann-Stieltjes integral):
  If $L : C[a,b] \to \mathbb{R}$ is linear and continuous, then $\exists\, \alpha \in BV[a,b]$ such that $\forall\, f \in C[a,b]$, $L(f) = \int_a^b f\, d\alpha$.