Lewin’s

“An introduction to mathematical grammar”

Chapter 1

An Introduction to Mathematical Grammar

1.1 THE ROLE OF PROOFS IN MATHEMATICS

Why do we have to prove theorems in mathematics? Among all the questions that students ask, this one is probably heard most often, and it is a good question which deserves a good answer. But answers to this question aren't so easy to find; and rightly or wrongly, many students would identify the answers they have received in their elementary mathematics courses with one or more of the following:

> Because I say what goes in this classroom and if you don't learn these proofs, you will fail.
>
> You don't! I was just mentioning this proof in case anyone happened to be interested, and since it is obvious that nobody is, let's get on with the only thing that really matters, and that is my instructions on how to do the homework problems.
>
> Look! I don't like this stuff any more than you do. But it's in the syllabus and so we have to do it.
>
> If you don't prove a theorem in mathematics, you won't really be sure that it is true.

Well, as you may have guessed, none of these answers comes close to providing the message which this book is meant to give you; and in fact, none of them represents true mathematical spirit. This statement applies even to the fourth answer, for it is quite wrong to say that we prove theorems in mathematics simply to make sure that they are true. If this were our only reason for proving theorems, why should more than one person have to prove them? If one person with a Ph.D. and an honest face announces that a theorem is true, then why shouldn't the rest of us just get on with our lives in the blissful knowledge that if we ever have to use that theorem, it will be there? And this question brings us back to the question we started with: Why do we need to prove theorems? We have two answers to suggest to you; the first takes a somewhat practical point of view, and the second reminds us that, after all, mathematics is an art form:

(1) A mathematical theorem represents much more than just a single statement. It represents a host of many statements, all of which can be deduced by roughly the same method of proof. A theorem therefore represents a level of mathematical understanding, for whoever understands its proof will command all the statements that can be deduced by the same method.

(2) Every mathematical theorem has two parts. The first part, which is called the *hypothesis,* contains the information which is assumed (or given); the second part, called the *conclusion,* is the part we have to prove. What the theorem really says is that *if* the hypothesis is assumed, *then* the conclusion must follow. Thus it is not the truth of the conclusion that is so important to us but, rather, the existence of a bridge between the hypothesis and the conclusion. The proof of the theorem is that bridge; and therefore, the proof of a theorem is the most important thing about it.

Let us look at some examples to illustrate the points we have been making.

(1) The opposite sides of a parallelogram have the same length. (See Figure 1.1.) If ever there were an obvious statement, then this simple fact of high school geometry must be it. Who could doubt that the opposite sides of a parallelogram must have the same length? Now cast your mind back a few years. Do you

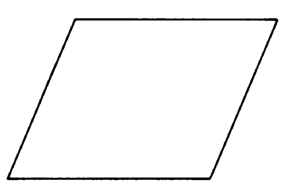

Figure 1.1

remember how this theorem is proved? We draw a diagonal which splits the parallelogram into two congruent triangles. Therefore, what the proof is really telling us is not so much that the opposite sides of a parallelogram must have the same length but, rather, that there is a bridge between this fact and the principles of congruent triangles. An understanding of this bridge enables us to deduce other theorems in geometry that are possibly less obvious.

(2) $$\sqrt[3]{\frac{9\sqrt{3}+5\sqrt{11}}{6\sqrt{3}}}+\sqrt[3]{\frac{9\sqrt{3}-5\sqrt{11}}{6\sqrt{3}}}=1.$$

At first sight this statement probably looks unbelievable, but now that we have told you that it is true, you can believe it. Why would we lie? Do you feel an urge to put the left side of this identity into your calculator? Go ahead and do it, but you will be wasting your time. Even before you begin, you know that your calculator must come up with the number 1. What point can there be in putting this expression into your calculator if you already know what the result will be? And after you are done, will you have any more understanding of the ideas that might lead to an identity like this one? Perhaps you would like to cube both sides. If so, prepare for a long, hard battle. And if you succeed, will you then understand how this identity was derived? The fact is that if you were to learn how one might *arrive* at this identity, then you would also know how to make many more of the same type; and you would *understand* why the identity has to hold.

(3) $$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots=\frac{\pi^2}{6}.$$

This example, like example (2), looks a little unbelievable, and once again, you could use your calculator to verify that the identity is at least approximately true. But once again, using your calculator wouldn't give you the slightest idea how you might arrive at such an identity. Surely, we need to ask just what it is about the infinite series on the left side that has anything to do with the number π. A proof of this identity can be found in Chapter 11.

In this book you will find many proofs you will need to know. Do not try to memorize them or they will rise against you and dominate you. Strive, instead, to understand them, and they will be your faithful servants. When studying a proof, do not be content with the knowledge of how each individual step follows from the one before it. Every proof has a *theme,* a master plan, which suggests what the individual steps should be. You have understood a proof only when you have looked into it deeply enough to perceive that theme, and you have understood a proof only when you feel able to write it down or explain it to others. As you progress and as you begin to understand the proofs this book contains, you will begin to command the ideas which underlie calculus.

1.2 THE DENIAL (NEGATION) OF A MATHEMATICAL SENTENCE

The *denial* (sometimes called the *negation*) of a given sentence is the statement that the given sentence is false. In the following examples we illustrate this notion.

(1) Suppose a given sentence says: *Everyone in this room speaks French.* In order for this statement to be false, there would have to be at least one person in this room who does not speak French. So the denial of the given sentence says: *At least one person in this room does not speak French.*

(2) Suppose a given sentence says: *The diagonals of a rectangle have the same length.* What this sentence really means in high school geometry is that every rectangle has the property that its diagonals have the same length. This example is therefore very similar to example (1), and its denial says: *There is at least one rectangle whose diagonals do not have the same length.*

(3) Suppose $f(x) = 32x/(1 + 4x^2)^2$ for $0 \leq x \leq 1$, and suppose that the given sentence says: *The maximum value of f is $3\sqrt{3}$.* This sentence really tells us two things. On the one hand, it tells us that $f(x) \leq 3\sqrt{3}$ for every $x \in [0, 1]$; and on the other hand, it tells us that there is at least one number $x \in [0, 1]$ for which $f(x) = 3\sqrt{3}$. The denial of this sentence therefore says: *Either there must exist a number $x \in [0, 1]$ for which $f(x) > 3\sqrt{3}$, or there is no number $x \in [0, 1]$ for which $f(x) = 3\sqrt{3}$.*

As you study mathematics, you should cultivate the habit of writing the denial of any sentence that seems difficult to understand. The general principle is that whenever you ask yourself what it means to say that a given statement is false, you will understand more clearly what it means to say that the statement is true. So, for example, in Chapter 6 you will see the definition of continuity of a function at a point: A function f is said to be continuous at a point a when. . . . When you come to Chapter 6, you will help yourself understand the definition of continuity if you ask yourself what it means to say that a given function *fails* to be continuous. The examples of Section 1.3.4 will help you learn how to ask this question, and you will have further practice writing denials in the exercises of Section 1.4.

1.3 SOME IMPORTANT SPECIAL WORDS USED IN MATHEMATICAL SENTENCES

1.3.1 The Words *and, or,* and *if.* The words *and, or,* and *if* are used in mathematics to combine two given sentences so as to make a single sentence. Consider, for example, the following two sentences p and q, where we take p to be the sentence "Helsinki is the capital of France," and q to be the sentence "All cats scratch." Among the ways we might combine these sentences are the following:

Helsinki is the capital of France, and all cats scratch.

Either Helsinki is the capital of France, or all cats scratch.

If Helsinki is the capital of France, then all cats scratch.

Looking at things more generally, let us suppose that p and q are any two given sentences. The following examples show some of the ways p and q might be combined.

(1) The sentence "p and q" means that both p and q are true. The denial of "p and q" is the statement that either p is false or q is false, or perhaps that both of them are false.

(2) The sentence "p or q" means that at least one of the two sentences p and q must be true (which includes the possibility that both of them are true). The denial of the sentence "p or q" is the statement that both of the sentences p and q must be false.

(3) The sentence "If p then q" means that in the event that p is true, then the sentence q must also be true. Note that the only way in which the sentence "If p then q" can be false is that p should be true and q should be false. The denial of "If p then q" therefore says that "p is true and q is false." The following list gives some of the equivalent ways in which we like to write the sentence "If p then q" in mathematics:

p implies q.

$p \Rightarrow q$.

q is implied by p.

$q \Leftarrow p$.

p is true only if q is true.

If q is false, then p is false.

Either p is false, or q must be true.

p is a sufficient condition for q.

q is a necessary condition for p.

(4) The sentence "p if and only if q," which is sometimes written as "p iff q," means that the sentences p and q are equivalent. In other words, "p iff q" means that either p and q are both true, or they are both false.

(5) The pair of sentences "p. Therefore q" is very often confused with "If p then q," but "p. Therefore q" says much more. When we say "p. Therefore q," we mean the following: "I know that p is true. I also know that if p is true, then q must be true, and I therefore conclude that q must be true." To help us understand the distinction between "If p then q" and "p. Therefore q," let us look at the following example: We take p to be the sentence "It is raining" and q to be the sentence "You will get wet." The sentence "If p then q" then says that "If it is raining, then you will get wet." More elaborately, "I don't know whether or not it is raining, but if it does happen to be raining, then you will get wet." On the other hand, "p. Therefore q" says "I know that it is raining and I therefore know that you will get wet."

1.3.2 Proving a Theorem Whose Statement Contains *and, or,* or *if*. As in the previous section, we shall suppose that p and q are sentences; and we shall suppose now that the sentences p and q have been combined into a single statement. How can we go about proving this single statement? The answer to this question depends on the way p and q have been combined, and in this section we suggest some possible approaches we might use.

First, suppose we want to prove a theorem whose conclusion is of the form "*p and q*." We need to show that both of the statements p and q are true, and we might do so, for instance, by showing first that p is true and then showing that q is true.

Second, if we want to prove a theorem whose conclusion is of the form "*p or q*," then we need to show that at least one of the two statements p and q is true. Among the many ways of showing this truth, the following three are quite common:

Assume that p is false, and use this assumption to show that q is true.

Assume that q is false, and use this assumption to show that p is true.

Assume that both p and q are false, and obtain a contradiction.

Third, in order to prove a theorem of the type "If p then q," we might use one of the following methods:

Assume that p is true, and use this assumption to show that q must be true.

Assume that q is false, and use this assumption to show that p is false.

Assume that p is true and q is false, and obtain a contradiction.

1.3.3 The Quantifiers *for every* and *there exists*. The phrases *for every* and *there exists* abound in mathematics. *For every* is called the *universal quantifier;* and depending upon the context, it sometimes appears simply as *every*, sometimes as *all*, and sometimes as the symbol $\forall$. *There exists* is called the *existential quantifier;* and depending upon the context, it sometimes appears as *there is, we can find, it is possible to find, there must be, there is at least one, some,* or as the symbol $\exists$. Notice that at least one of these quantifiers appears in each of the examples of the next section and in the exercises that follow.

1.3.4 Some Examples. The examples of this section are designed to help you understand the use of the words *and*, *or*, *if*, *for every*, and *there exists*. Read them alongside their denials, and notice how often an awareness of the denial will help you understand the given statement.

(1) All cats scratch.
Denial. Not all cats scratch. An alternate form of this denial is, "There exists a cat which does not scratch."

(2) There exists a cat which scratches.
Denial. All cats do not scratch.

Warning. Do *not* confuse these two examples. Outside mathematics, it is quite common to hear people saying "All cats do not scratch" when what they really mean is "Not all cats scratch." It's wrong to confuse these sentences under any circumstances, but even if you do it outside mathematics, *don't do it here!*

(3) Some cats scratch. This sentence is example (2) again.

(4) All cats scratch and some dogs bite.
Denial. At least one cat does not scratch or no dogs bite.

(5) If some cats scratch, then all dogs bite.
Denial. Some cats scratch, and some dogs do not bite.

(6) Either some cats scratch, or if all dogs bite, then some birds sing.
Denial. No cats scratch, and all dogs bite, and no birds sing.

(7) Once upon a time there was a princess.
Denial. There has never been a princess.

(8) Once upon a time there lived a beautiful princess.
Denial. No princess who has ever lived has been beautiful.

(9) No Irishman has ever been at a loss for words.
Denial. At least one Irishman has on at least one occasion been at a loss for words.

(10) No one has ever seen an Englishman who is not carrying an umbrella.
Denial. At least one person has on at least one occasion seen an Englishman who is not carrying an umbrella.

(11) For every positive number x the number $x - 1$ is also positive.
Denial. There exists a positive number x such that $x - 1 \leq 0$.

(12) For every number x there exists a number y such that $y > x$.
Denial. There exists a number x such that for every number y we have $y \leq x$.

(13) There is no least positive number.
Denial. There exists a positive number x such that for every positive number y we have $y \geq x$.

(14) For every number x either $x^2 > 1$ or $x \leq 1$.
Denial. There exists a number x such that $x^2 \leq 1$ and $x > 1$.

The last few examples concern two functions f and g which we assume have been given.

(15) Whenever $x > 50$, we have $f(x) = g(x)$.
Denial. There exists a number x such that $x > 50$ and $f(x) \neq g(x)$.

(16) There exists a number w such that $f(x) = g(x)$ for all numbers $x > w$.
Denial. For every number w there exists a number $x > w$ such that $f(x) \neq g(x)$.

(17) For every number x there exists a number $\delta > 0$ such that for every number t satisfying the condition $|x - t| < \delta$, we have $|f(x) - f(t)| < 1$.
Denial. There exists a number x such that for every positive number δ it is possible to find a number t such that $|x - t| < \delta$ and $|f(x) - f(t)| \geq 1$.

(18) There exists a number $\delta > 0$ such that for every pair of numbers x and t satisfying the condition $|x - t| < \delta$, we have $|f(x) - f(t)| < 1$.
Denial. For every positive number δ there exists a pair of numbers x and t such that $|x - t| < \delta$ and $|f(x) - f(t)| \geq 1$.

(19) For every number $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every pair of numbers x and t satisfying the conditions $|x - 2| < \delta$ and $|t - 2| < \delta$, we have $|f(x) - f(t)| < \varepsilon$.
Denial. There exists a positive number ε such that for every positive number δ it is possible to find a pair of numbers x and t satisfying the conditions $|x - 2| < \delta$ and $|t - 2| < \delta$, and $|f(x) - f(t)| \geq \varepsilon$.

(20) For every number $\varepsilon > 0$ and for every number x, there exists a number $\delta > 0$ such that for every number t satisfying $t \neq x$ and $|x - t| < \delta$, we have $|f(t) - g(t)| < \varepsilon$.
Denial. There exists a positive number ε and there exists a number x such that for every positive number δ it is possible to find a number $t \neq x$ such that $|x - t| < \delta$ and $|f(t) - g(t)| \geq \varepsilon$.

(21) For every positive number ε there exists a positive number δ such that for every pair of numbers x and t satisfying the conditions that $x \neq t$ and $|x - t| < \delta$, we have $|f(x) - f(t)| < \varepsilon$.
Denial. There exists a positive number ε such that for every positive number δ it is possible to find numbers x and t such that $x \neq t$ and $|x - t| < \delta$, and $|f(x) - f(t)| \geq \varepsilon$.

1.4 EXERCISES

In the following exercises[1] f and g are given functions and A and B are given sets of real numbers. In each exercise, write the denial of the given sentence in a form that is pleasant to read and uses proper English.

■ **1.** If what you said yesterday is correct, then Jim has red hair.

■ **2.** Either you take me for a fool, or you must be a fool yourself.

■ **3.** He walked into my office this morning, told me a pack of lies, and punched me on the nose.

■ **4.** All that glitters is gold.

■ **5.** You are right and I am wrong.

■ **6.** You are right if and only if I am wrong.

■ **7.** Either there exists a cat that does not scratch, or everyone in this room is a liar.

■ **8.** I dream when I sleep. (Lewis Carroll)

■ **9.** Nobody is worth listening to on military subjects, unless he can remember the battle of Waterloo. (Lewis Carroll)

■ **10.** Some of us are out of breath and all of us are fat. (Lewis Carroll)

■ **11.** Someone in this room is smoking.

■ **12.** Fifty percent of the people in this room are smoking.

■ **13.** It is with regret that I inform you that someone in this room is smoking.

■ **14.** There exists a number x such that for every number $u > x$ we have $f(u) > g(u)$.

■ **15.** For all numbers u and v which satisfy $u > 50$ and $v > 50$, we have $|f(u) - f(v)| < 2$.

■ **16.** There exists a number p such that for all numbers u and v which satisfy $u > p$ and $v > p$, we have $|f(u) - f(v)| < 2$.

■ **17.** For every number $\varepsilon > 0$ and for all numbers x and t satisfying $x > 7$ and $t > 7$, we have $|f(x) - f(t)| < \varepsilon$.

■ **18.** For every number $\varepsilon > 0$ there exists a number p such that for all numbers x and t satisfying $x > p$ and $t > p$, we have $|f(x) - f(t)| < \varepsilon$.

■ **19.** There exists a number p such that for every number $\varepsilon > 0$ and for all numbers x and t satisfying $x > p$ and $t > p$, we have $|f(x) - f(t)| < \varepsilon$.

[1] As explained in the preface, the symbol ■ is used to designate an exercise for which a solution (or partial solution) has been provided.

▪**20.** If a function h is continuous on $(0, 1)$, then there must exist a number x in $(0, 1)$ such that h is differentiable at x.

▪**21.** There exists a number w such that for every member x of A we have $x < w$.

▪**22.** Whenever x is a member of A and y is a member of B, we have $x < y$.

▪**23.** Whenever x and y are members of A, either $x = y$ or $|x - y| \geq 1$.

▪**24.** For every positive number ε it is possible to find two members x and y of A such that $x \neq y$ and $|x - y| < \varepsilon$.

▪**25.** If P and Q are any two sets of numbers, and if for every member x of P and every member y of Q, we have $x < y$, then it is possible to find a number w such that whenever x is a member of P and y is a member of Q, we have $x \leq w \leq y$.

1.5 A GUIDE TO THE PROPER INTRODUCTION OF MATHEMATICAL SYMBOLS

To help us appreciate the need for this introduction, let's ask a little question about elementary algebra:

Is it true that $(x + y)^2 = x^2 + y^2$?

Perhaps the answer "no" is hovering on your lips. If so, you are being a little hasty; for is it not true that $(3 + 0)^2 = 3^2 + 0^2$? As you can see, the truth or falsity of the equation $(x + y)^2 = x^2 + y^2$ depends upon precisely which numbers x and y we are talking about. Therefore, since this question says nothing about what the numbers x and y are, we must conclude that the question is meaningless as it stands.

Now let's take a look at a few of the ways this question might have been asked meaningfully:

Is it true that there exist numbers x and y such that $(x + y)^2 = x^2 + y^2$? Yes!

Is it true that for every number x there exists a number y such that $(x + y)^2 = x^2 + y^2$? Yes!

Is it true that there exists a number x such that for every number y we have $(x + y)^2 = x^2 + y^2$? Yes!

Is it true that for all numbers x and y we have $(x + y)^2 = x^2 + y^2$? No!

Notice how in each of these four meaningful variations of the question, the symbols x and y were introduced by one or other of the quantifiers *for every* and *there exists*. And in the same way, if you look back at the examples and exercises of the preceding sections, you will see that, with the exception of the few symbols that were declared as having been given at the beginning of the section, every symbol was carefully introduced when it first appeared, with one or other of the two quantifiers. Perhaps you have also noticed that as one moves from a sentence to its denial, the quantifiers *for every* and *there exists* change places. And finally, you may have noticed that it is important to know precisely where in the sentence a given symbol is introduced. For example, compare the following two sentences:

(1) For every positive number x there exists a positive number y such that $y < x$.

(2) There exists a positive number y such that for every positive number x we have $y < x$.

Although these statements may look similar, they do not say the same thing. As a matter of fact, (1) is true and (2) is false. If you look back at the sentences in Sections 1.3.4 and 1.4, you will see a number of pairs of similar-looking sentences whose meanings are quite different because of differences in the order of appearance of the symbols. So the message of this section is as follows: For a piece of mathematical writing to make sense, all symbols must be properly introduced by one or the other of the two quantifiers, and each symbol must be introduced in the right place—not too early and not too late.

1.6 HOW TO PROVE A THEOREM THAT SAYS "THERE EXISTS . . ."

We shall begin by looking at a nonmathematical example. You are standing in a room full of people, and you are asked to demonstrate that there is at least one bald man in the room. One thing you might do is go from man to man and in each case look at his head, until you have found a bald man. If your search turns up at least one bald man, you can *choose* one of the bald men you have found as an *example* of a bald man in the room; and by *giving an example*, you have proved that a bald man exists.

As a simple example of this technique in mathematics, we shall prove the following easy theorem:

There exists a prime number greater than 20.

To prove this theorem, we shall *choose* 71 as an example of a prime number greater than 20. This example proves the theorem.

Now let's look at this kind of statement in general. Suppose we wish to prove that a certain set A is not empty; in other words, we want to prove that a member of A must exist. For example, A might be the set of bald men in a given room or the set of prime numbers greater than 20. If we happen to know of an example of a member of A, then we can be sure that the set A is not empty.

We mention finally that giving an example is not the *only* method of proving existence. Suppose you were asked to verify that someone in a given room is smoking. You wouldn't have to see and identify a smoker in order to know that a smoker exists. All you would have to do is try to breathe. In mathematics, too, we

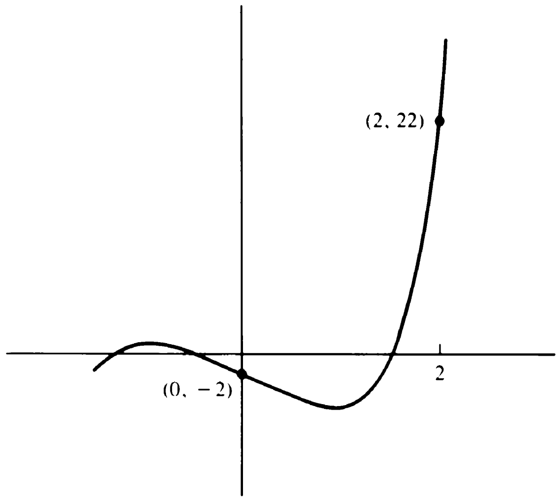

Figure 1.2

often prove existence by this sort of indirect method. As an example, we shall sketch a proof of the following statement:

There exists a number x between 0 and 2 such that $x^5 - 4x - 2 = 0$.

Some of the details of this proof will have to wait until we reach Chapter 6. But we can give a sketch of the proof here. For every real number x, define $f(x) = x^5 - 4x - 2$. Note that $f(0) < 0$ and $f(2) > 0$. But f, being a polynomial, must be *continuous*. And as you will see when you reach Section 6.8, if a function is continuous on an interval and has a negative value at one endpoint and a positive value at the other, there must be a point in the interval at which the function is zero. (See Figure 1.2.)

So even though we do not know of an example of a number x between 0 and 2 at which $f(x) = 0$, we know that such a number exists. As we have said, giving an example isn't the only way to prove existence; but it is often the easiest way, and so we give an example whenever we can.

1.7 HOW TO USE A THEOREM THAT SAYS "THERE EXISTS . . ."

If we are *given* (or if we have previously proved) that a certain set A is nonempty, and if we need a member of that set, then we may *choose* a member of the set as an example. As in the previous section, we shall illustrate this idea by looking first at a nonmathematical example: You are standing in a room full of people, and the person next to you collapses. You call, "Is there a doctor in the house?" Then having established that a doctor exists, you *choose* one of the doctors to help the person who is ill.

Now let us apply this technique to mathematics. As you may know, it is important in calculus to have a function f with the property that $f'(x) = 1/x$ for every $x > 0$. Having proved that such functions exist, we *choose* one of them [the one we choose is the one satisfying $f(0) = 1$]; and we call this function the *natural log function*. You will find the details of this procedure in Chapter 9.

When you are writing a mathematical proof, do not make the common mistake of thinking that just because a certain set A is known to be nonempty, a member of A has automatically been chosen for you. No member has been chosen until you choose it. For example, consider the following statement:

There exists a number x such that $0 < x < 1$.

Does this sentence refer to any particular number x in $(0, 1)$? No, it doesn't! All this sentence says is that the interval $(0, 1)$ is nonempty, and we would be quite wrong to follow it with statements about x as if there were some x which had been introduced. For example, we would be incorrect if we wrote the following sentences:

There exists a number x such that $0 < x < 1$.

Clearly, $x^2 < x$.

On the other hand, we can legitimately say:

There exists a number x such that $0 < x < 1$.

Furthermore, for every $x \in (0, 1)$ we have $x^2 < x$.

Alternatively, we could say:

There exists a number x such that $0 < x < 1$.

Choose a number $x \in (0, 1)$. Note that $x^2 < x$.

Note that when we say that there exists a number $x \in (0, 1)$, there is nothing special about the symbol x. This statement could just as well have been written in one of the following equivalent forms:

There exists a number t such that $0 < t < 1$.

There exists a number p such that $0 < p < 1$.

There exists a number in the interval $(0, 1)$.

1.8 HOW TO PROVE A THEOREM THAT SAYS "FOR EVERY . . ."

As in the previous two sections, we shall begin by looking at a nonmathematical example. Suppose you are standing in a room full of people and you are asked to prove that every man in the room is bald. It would surely not be good enough to produce just one bald man! You have to examine the head of every man in the room to make sure that he is bald. Should you find even one nonbald man, the proposition you are trying to prove is false. But if, after you have examined every man, you can say that they were all bald, then your proof has succeeded.

The analogue of this kind of statement in mathematics is a statement that says that all the members of a given set A must have a certain property. As an example of such a statement, we shall look again at sentence (12) of Section 1.3.4:

For every number x there exists a number y such that $y > x$.

Unfortunately, one can't prove this mathematical theorem by an exact analogue of the method we have suggested for proving that all the men in a room are bald. The trouble is that there are infinitely many numbers; and even if we were to spend the rest of our lives checking numbers one at a time to make sure that for every number x there exists a number y such that $y > x$, we would eventually die, leaving the unproved theorem as a legacy. This is where mathematicians use the word *let*. To prove the theorem, we proceed as follows:

Let x be any real number.

What this sentence really means is: "I don't know if there are any real numbers to call x and I don't care. I don't need any real numbers; but in case there are any, let x be an arbitrary one of them, coming without any restrictions, so that anything I might be able to prove about x would apply just as well to any other number. In other words, let x be an arbitrary number come to *challenge* me to prove that there exists a number y such that $y > x$."

Now that we have this challenger x in our hands, we have to prove that there exists a number y such that $y > x$. For this purpose we shall use the fact that $x + 1 > x$. We write:

Define $y = x + 1$.

Since this choice has provided us with an example of a number $y > x$, the proof is complete.

We end this section with an example of a proof that is a little different from the one just given. This time we shall prove the theorem that says that

$\sqrt{2}$ is irrational.

What this theorem really means is that for every pair of natural numbers m and n we have $m/n \neq \sqrt{2}$, or in other words, $m^2 \neq 2n^2$. The technique described above therefore suggests that in order to prove this statement, we should begin as follows:

Let m and n be natural numbers.

In the proof that follows we use a variation on this theme. First, we observe that if $\sqrt{2}$ were rational, then one could write $\sqrt{2}$ in the form m/n, where m and n are natural numbers which have no common factor (except 1). We now begin our proof of the irrationality of $\sqrt{2}$ by saying:

Let m and n be natural numbers with no common factor, and to obtain a contradiction, assume that $m^2 = 2n^2$.

The equation $m^2 = 2n^2$ implies that n^2 must be a factor of m^2. And since the numbers m^2 and n^2 have no common factor, it follows that $n = 1$. Therefore, $m^2 = 2$. Since there is clearly no natural number m such that $m^2 = 2$, we have reached the desired contradiction.

To sum up this section, a major technique used in mathematics to prove a statement that says that all members of a given set S must have a certain property, is to begin the proof with the following words:

Let x be a member of the set S.

1.9 HOW TO USE A THEOREM THAT SAYS "FOR EVERY . . ."

Let's go back to that room full of people. Suppose you have been reliably informed that all bald men are honest. And suppose you happen to be talking to a certain bald man, and this man wants to sell you oil stock. Then you may safely invest your money. Understand that the only way that the information that all bald men are honest can be of any use to you is that there should be certain bald men of special interest to you. Perhaps you are talking to them. Perhaps you are related to them. But one way or another, there are certain bald men of special interest to you, and you would like to know that *they* are honest. You may come to this conclusion because of your knowledge that *all* bald men are honest.

The mathematical analogue of this idea is almost identical. Suppose you happen to know that all the members of a given set S have a certain property, and suppose you have a special interest in some particular members of the set. Perhaps you are talking to these members, or perhaps you are related to them. One way or another, you have a special interest in certain members of S, and you would like to know that these particular members have the property mentioned. You may conclude that they do because of your knowledge that *all* members of S have the desired property.

Do not confuse the problem of *proving* that all the members of a given set S have a certain property with the question we are discussing here. As you saw in Section 1.8, when you want to *prove* that every member of a set S has a certain property, then you write, "Let x be a member of the set S." It would be a mistake to write this sentence when you know that all members of a set S have some property, and you want to *use* this fact. Once again, knowledge that all members of a set S have a certain property can only be useful to you when you have a special interest in particular members of S.

This ends our grammar lesson, and you are now ready to enter Chapter 2, where you will begin your study of mathematical analysis. As you do so, be guided by the principles you have learned here. Your success in the coming chapters will be dependent on how well you learn to implement these principles.