1. CLAIM: $f^{-1}[A \setminus B] = f^{-1}[A] \setminus f^{-1}[B]$ (i.e., $x \in f^{-1}[A \setminus B] \Leftrightarrow x \in f^{-1}[A] \setminus f^{-1}[B]$)

PROOF: $x \in f^{-1}[A \setminus B] \Leftrightarrow f(x) \in A \setminus B$
$\Leftrightarrow f(x) \in A$ AND $f(x) \notin B$
$\Leftrightarrow x \in f^{-1}[A]$ AND $x \notin f^{-1}[B]$
$\Leftrightarrow x \in f^{-1}[A] \setminus f^{-1}[B]$ $\blacksquare$

2. CLAIM: $f^{-1}[A \cup B] = f^{-1}[A] \cup f^{-1}[B]$ (i.e., $x \in f^{-1}[A \cup B] \Leftrightarrow x \in f^{-1}[A] \cup f^{-1}[B]$)

PROOF: $x \in f^{-1}[A \cup B] \Leftrightarrow f(x) \in A \cup B$
$\Leftrightarrow f(x) \in A$ OR $f(x) \in B$
$\Leftrightarrow x \in f^{-1}[A]$ OR $x \in f^{-1}[B]$
$\Leftrightarrow x \in f^{-1}[A] \cup f^{-1}[B]$ $\blacksquare$

3. CLAIM: $f^{-1}[A \cap B] = f^{-1}[A] \cap f^{-1}[B]$ (i.e., $x \in f^{-1}[A \cap B] \Leftrightarrow x \in f^{-1}[A] \cap f^{-1}[B]$)

PROOF: $x \in f^{-1}[A \cap B] \Leftrightarrow f(x) \in A \cap B$
$\Leftrightarrow f(x) \in A$ AND $f(x) \in B$
$\Leftrightarrow x \in f^{-1}[A]$ AND $x \in f^{-1}[B]$
$\Leftrightarrow x \in f^{-1}[A] \cap f^{-1}[B]$ $\blacksquare$

4. SUPPOSE $f: X \to Y$.

CLAIM: $\forall\, B \subset Y$, $f[f^{-1}[B]] \subset B$

PROOF: LET $B \subset Y$. (NEED TO SHOW $y \in f[f^{-1}[B]] \Rightarrow y \in B$)

THEN $y \in f[f^{-1}[B]] \Leftrightarrow \exists\, x \in f^{-1}[B]$ WITH $y = f(x)$
$\Leftrightarrow \exists\, x$ WITH $f(x) \in B$ AND $y = f(x)$

TAKE SUCH AN x. THEN $y = f(x) \in B$ $\blacksquare$

CLAIM: $\left(\forall\, B \subset Y,\ f[f^{-1}[B]] = B\right) \Leftrightarrow f$ IS SURJECTIVE

PROOF $\Rightarrow$: SUPPOSE THAT $\forall\, B \subset Y$, $f[f^{-1}[B]] = B$

(NEED TO SHOW THAT f IS SURJECTIVE, I.E., $f[X] = Y$)

LET $B = Y$. THEN $B \subset Y$, SO $f[f^{-1}[B]] = B$

I.E., $f[f^{-1}[Y]] = Y$.

BUT $f^{-1}[Y] = X$, SO $f[X] = Y$

(I.e., $x \in f^{-1}[Y] \Leftrightarrow x \in X$
$f(x) \in Y \Leftrightarrow x \in X$)

PROOF $\Leftarrow$: SUPPOSE THAT f IS SURJECTIVE, I.E., $f[X] = Y$.

THEN $Y \subset f[X]$, SO $y \in Y \Rightarrow y \in f[X] \Rightarrow \exists\, x \in X$ WITH $f(x) = y$

(NEED TO SHOW THAT $\forall\, B \subset Y$, $f[f^{-1}[B]] = B$)

LET $B \subset Y$. WE SHOWED ABOVE THAT $f[f^{-1}[B]] \subset B$,

SO IT SUFFICES TO SHOW THAT $B \subset f[f^{-1}[B]]$

(I.E., $b \in B \Rightarrow b \in f[f^{-1}[B]]$)

LET $b \in B \subset Y$. BY HYPOTHESIS, $\exists\, x \in X$ WITH $f(x) = b$.

TAKE SUCH AN x.

THEN $f(x) = b \in B$, SO BY DEFINITION, $x \in f^{-1}[B]$.

(of $f^{-1}[\ldots]$)

BUT THEN $x \in f^{-1}[B]$ AND $f(x) = b$,

SO BY DEFINITION, $b \in f[f^{-1}[B]]$ $\blacksquare$

(of $f[\ldots]$)

9. FOR BOTH PARTS, TAKE $f: \{a, b\} \to \{1, 2\}$, $f(a) = 1 = f(b)$.

(i) $A, B \subset X$ WITH $f[A \cap B] \neq f[A] \cap f[B]$

LET $A = \{a\}$, $B = \{b\}$. THEN $f[A \cap B] = f[\emptyset] = \emptyset$,

WHILE $f[A] \cap f[B] = \{1\} \cap \{1\} = \{1\}$.

(ii) $A \subset X$, $f[X \setminus A] \neq f[X] \setminus f[A]$

LET $A = \{a\}$. THEN $f[X \setminus A] = f[\{b\}] = \{1\}$,

WHILE $f[X] \setminus f[A] = \{1\} \setminus \{1\} = \emptyset$.

5. SUPPOSE $f: X \to Y$.

CLAIM: $\forall A \subset X$, $A \subset f^{-1}[f[A]]$

PROOF: LET $A \subset X$. (NEED TO SHOW $a \in A \Rightarrow a \in f^{-1}[f[A]]$)

$a \in A \Rightarrow f(a) \in f[A]$.

SO, BY DEFINITION, $a \in f^{-1}[f[A]]$ ■
(OF $f^{-1}[\ldots]$)

CLAIM: ($\forall A \subset X$, $A = f^{-1}[f[A]]$) $\Leftrightarrow$ f IS INJECTIVE

PROOF $\Rightarrow$: SUPPOSE THAT $\forall A \subset X$, $A = f^{-1}[f[A]]$.
(NEED TO SHOW THAT f IS INJECTIVE, I.E., $f(x) = f(x') \Rightarrow x = x'$)

SUPPOSE THAT $f(x) = f(x')$.

THEN $f[\{x\}] = \{f(x)\} = \{f(x')\} = f[\{x'\}]$

SO $f^{-1}[f[\{x\}]] = f^{-1}[f[\{x'\}]]$.

THUS, BY HYPOTHESIS, $\{x\} = \{x'\}$, SO $x = x'$ ✓

PROOF $\Leftarrow$: SUPPOSE THAT f IS INJECTIVE, I.E., $f(x) = f(x') \Rightarrow x = x'$
(NEED TO SHOW THAT $\forall A \subset X$, $A = f^{-1}[f[A]]$)

LET $A \subset X$. WE SHOWED ABOVE THAT $A \subset f^{-1}[f[A]]$,
SO IT SUFFICES TO SHOW THAT $f^{-1}[f[A]] \subset A$
(I.E., $x \in f^{-1}[f[A]] \Rightarrow x \in A$)

SUPPOSE THAT $x \in f^{-1}[f[A]]$.

THEN BY DEFINITION, $f(x) \in f[A]$,
SO $\exists\, a \in A$ WITH $f(x) = f(a)$.

TAKE SUCH AN a.

THEN, BY HYPOTHESIS, $f(x) = f(a) \Rightarrow x = a$, SO $x = a \in A$ ✓ ■

6. SUPPOSE THAT $f: X \to Y$. (LET'S SET THIS UP LIKE #4 AND #5)

CLAIM: $\forall A, B \subset X$, $f[A \cap B] \subset f[A] \cap f[B]$

PROOF: LET $A, B \subset X$. (NEED TO SHOW THAT $y \in f[A \cap B] \Rightarrow y \in f[A] \cap f[B]$)

$y \in f[A \cap B] \Rightarrow \exists\, x \in A \cap B$ WITH $y = f(x)$

TAKE SUCH AN x. THEN $x \in A$ AND $y = f(x)$, SO $y \in f[A]$;
SIMILARLY, $x \in B$ AND $y = f(x)$, SO $y \in f[B]$.

THUS, BY DEFINITION, $y \in f[A] \cap f[B]$ ■

CLAIM: ($\forall A, B \subset X$, $f[A \cap B] = f[A] \cap f[B]$) $\Leftrightarrow$ f IS INJECTIVE

PROOF $\Rightarrow$: SUPPOSE THAT $\forall A, B \subset X$, $f[A \cap B] = f[A] \cap f[B]$.
(NEED TO SHOW THAT f IS INJECTIVE, I.E., $f(x) = f(x') \Rightarrow x = x'$)

SUPPOSE THAT $f(x) = f(x')$, AND LET $A = \{x\}$, $B = \{x'\}$.

THEN, BY HYPOTHESIS, $f[\{x\} \cap \{x'\}] = f[\{x\}] \cap f[\{x'\}]$
$= \{f(x)\} \cap \{f(x')\}$
$= \{f(x)\}$ BECAUSE $f(x) = f(x')$.

IT MUST THEN BE TRUE THAT $x = x'$,
FOR OTHERWISE $\emptyset = f[\emptyset] = f[\{x\} \cap \{x'\}] = \{f(x)\}$ ✓

PROOF $\Leftarrow$: SUPPOSE THAT f IS INJECTIVE, I.E., $f(x) = f(x') \Rightarrow x = x'$.
(NEED TO SHOW THAT $\forall A, B \subset X$, $f[A \cap B] = f[A] \cap f[B]$)

LET $A, B \subset X$. WE SHOWED ABOVE THAT $f[A \cap B] \subset f[A] \cap f[B]$,
SO IT SUFFICES TO SHOW THAT $f[A] \cap f[B] \subset f[A \cap B]$.
(I.E., $y \in f[A] \cap f[B] \Rightarrow y \in f[A \cap B]$)

SUPPOSE $y \in f[A] \cap f[B]$

THEN $y \in f[A] \Rightarrow \exists\, a \in A$ WITH $y = f(a)$, AND SIMILARLY,
$y \in f[B] \Rightarrow \exists\, b \in B$ WITH $y = f(b)$.

NOW, $f(a) = y = f(b)$, SO BY HYPOTHESIS, $a = b$. LET $x = a = b$.

THEN $x \in A \cap B$ AND $y = f(x)$, SO, BY DEFINITION, $y \in f[A \cap B]$ ✓ ■